

Partial Differential Equations
In Engineering Problems

KENNETH S. MILLER

PROPERTY OF

*The
University of
Michigan
Libraries*

1817

ARTES SCIENTIA VERITAS

Partial Differential Equations
In Engineering Problems

Partial Differential Equations *In Engineering Problems*

BY

KENNETH S. MILLER, Ph.D.

Associate Professor of Mathematics
New York University

ENGLEWOOD CLIFFS, N. J.

PRENTICE-HALL, INC.,

Institute of
Sci. & Tech.

QA
374
• M65

**ALL RIGHTS RESERVED BY PRENTICE-HALL, INC., 70 FIFTH
AVE., NEW YORK. NO PART OF THIS BOOK MAY BE RE-
PRODUCED, BY MIMEOGRAPH OR ANY OTHER MEANS, WITHOUT
PERMISSION IN WRITING FROM THE PUBLISHER.**

First printing.....October , 1953
Second printing.... February , 1959
Third printing..... July , 1961
Fourth printing..... March , 1963

Printed in the United States of America

65040-C

PREFACE

We have attempted in this book to present the subject of partial differential equations at a level which can be appreciated and used by the engineering student. Only an elementary knowledge of ordinary differential equations is a prerequisite to its understanding. The guiding light throughout the book is the practical solution of partial differential equations, little attention being paid to questions of existence, uniqueness, and convergence. It is felt that students (upper classmen or beginning graduate students) are not interested in such considerations, and that these topics might obscure the fundamental techniques rather than add to the clarity of the exposition. Furthermore, since all illustrative problems are based on physical systems, physical reasoning can be used to guarantee intuitively the existence of a solution and make plausible the fact that the function we formally obtain is actually a true solution.

In Chapter I we derive, in typical engineering fashion, some of the more common partial differential equations associated with such phenomena as vibration, heat flow, electricity, and elasticity. The second chapter is devoted to an exposition of fundamental facts concerning Fourier series. These ideas are, of course, useful in themselves; but their main value from our point of view is their use in solving partial differential equations. In Chapter III we apply the techniques of Fourier analysis developed in Chapter II to the equations derived in Chapter I and solve a number of equations by the method of separation of variables. Certain problems, for example those involving nonperiodic functions, are not amenable to the method of Fourier series, hence in Chapter IV we extend the Fourier method to arrive at the Fourier integral, which enables us to solve a wider class of problems. The Fourier integral also leads to the "transform method" of solving partial differential equations. In Chapter V we return to the method of separation of variables in conjunction with changes in the coordinate systems, and are led to certain ordinary differential equations of practical and theoretical importance, namely,

Legendre's, Bessel's, and Mathieu's equations. A digression must be made to consider various elementary properties of the Legendre, Bessel, and Mathieu functions. Armed with this information we can solve, by the method of separation of variables, the partial differential equations which gave rise to these special ordinary differential equations. It seems appropriate in a book of this kind to consider certain elementary aspects of the general theory. Chapter VI gives the student some insight into the general structure of differential operators.

Other methods of solution of partial differential equations, such as the Laplace transform and numerical procedures, have not been included in this book, both in order to limit its size and scope and because excellent texts on these particular methods are available.

Part of the material covered in this book has been the basis of courses taught by Professors M. G. Salvadori and R. J. Schwarz for the last ten years in the School of Engineering, Columbia University, and by the author in the College of Engineering, New York University. I am indebted to Professors Salvadori and Schwarz, who read the entire manuscript and offered many valuable suggestions, and to my father, Mr. W. A. Miller, for the preparation of the many excellent diagrams.

K. S. M.

CONTENTS

I.	Derivation of Partial Differential Equations 1. Introduction. 2. The vibrating string. 3. Boundary conditions. 4. The vibrating membrane. 5. Longitudinal vibrations of a bar. 6. Transverse vibrations of a beam. 7. Heat flow in one dimension. 8. The three-dimensional heat equation. 9. The transmission line equations. 10. Potential. 11. The equation of continuity. 12. Tidal waves in an incompressible fluid. 13. The equations of elasticity. 14. Additional equations.	1
II.	Fourier Series 15. Preliminary remarks. 16. Some integral formulas. 17. Calcula- tion of the coefficients. 18. Recapitulation. 19. Example of a Fourier series. 20. Change of interval. 21. Example of change of interval. 22. Even and odd functions. 23. Examples of expansion of even and odd functions. 24. Half Fourier intervals. 25. Examples of even and odd prolongations. 26. Points of discontinuity. 27. Trig- onometric approximations. 28. Trigonometric identities. 29. Fourier series expansion in $(-L, L)$. 30. Example of a Fourier series in $(-L, L)$. 31. Even and odd functions in $(-L, L)$. 32. Examples of even and odd functions in $(-L, L)$. 33. Prolongation of functions in $(-L, L)$. 34. Phase angles. 35. Example of a Fourier series involving a phase angle. 36. Complex Fourier series. 37. Example of a complex Fourier series. 38. Double trigonometric series. 39. Example of a double Fourier series. 40. Mathematical conditions on Fourier series expansions.	45
III.	Separation of Variables 41. Introduction. 42. Solution of the vibrating string problem. 43. Alternate form for the solution of the vibrating string. 44. An example of the wave equation. 45. Temperature distribution in a slab. 46. Example of the heat equation. 47. Voltage in a transmission line. 48. Example of the transmission line equation. 49. Non- homogeneous boundary conditions. 50. Temperature distribution in a semicircular plate. 51. Transverse vibrations of a beam. 52. The	93

vibrating membrane. 53. Nonhomogeneous equations. 54. The heat equation with radioactive decay. 55. The general case. 56. A trial function. 57. A trial function method based on Fourier series.

IV. The Fourier Integral 142

58. A heat flow problem. 59. The sine and cosine transforms. 60. Continuation of the solution of Section 58. 61. A digression on integration. 62. Completion of the solution of Section 58. 63. A numerical example. 64. Vibrations of an infinite string. 65. The Fourier transform. 66. Examples of Fourier transforms. 67. Properties of Fourier transforms. 68. Transform of the heat equation. 69. Transform of the wave equation. 70. Mathematical conditions on the Fourier integral.

V. Legendre, Bessel, and Mathieu Functions 175

71. Some important ordinary differential equations. 72. Legendre's equation. 73. Bessel's equation. 74. Mathieu's equation. 75. Orthogonal functions. 76. The gamma function. 77. Solution of Legendre's equation. 78. Orthogonality of the Legendre polynomials. 79. Generating function for the Legendre polynomials. 80. Completion of the solution of Section 72. 81. Solution of Bessel's equation. 82. Properties of the Bessel functions. 83. Orthogonality of the Bessel Functions. 84. Completion of the solution of Section 73. 85. Solution of Mathieu's equation. 86. Orthogonality of the Mathieu functions. 87. Completion of the solution of Section 74. 88. A unifying principle.

VI. Properties of Second-Order Partial Differential Equations 234

89. Introduction. 90. Transformation of variables. 91. Canonical forms. 92. The hyperbolic equation. 93. The wave equation and the Cauchy problem. 94. The homogeneous equation of hyperbolic type. 95. The parabolic equation. 96. The elliptic equation.

References 249

Index 251

Chapter I

DERIVATION OF PARTIAL DIFFERENTIAL EQUATIONS

1. Introduction

Partial differential equations arise in many physical problems and, as in the case of ordinary differential equations [25],* a great many unrelated physical phenomena give rise to the same equation if the variables and constants are appropriately interpreted. In setting up the partial differential equations governing certain physical phenomena we shall restrict ourselves to elementary equations requiring little knowledge of the specialized fields of engineering. However, we shall assume that the reader has some familiarity with the fundamentals of mechanics, heat, electricity, and fluid mechanics.

The list of equations derived in this book is by no means exhaustive : it is merely representative. At the end of this chapter a list of other important equations is given, with references to books containing their derivations. This category of equations requires a more specialized knowledge of the various branches of engineering. The equations considered in the text might be called the "classical partial differential equations of mathematical physics." They will be derived in typical engineering fashion, that is, with more appeal to physical and qualitative reasoning than to mathematical rigor.

After the chapter on derivations (which may be omitted by any reader who is interested in the *solution* but not in the origin of the equations), we shall develop in Chapter II a powerful tool, Fourier series, which will enable us to solve some of the equations of

* Numbers in square brackets refer to the references at the end of this book.

Chapter I. In Chapter III the methods of Fourier series will be applied to obtain explicit solutions for a wide variety of physical problems. Later chapters will deal with various extensions of the tools of Chapter II which are necessary to solve wider classes of partial differential equations than those amenable to solution by Fourier series.

It will be necessary to recall a formula from the calculus (see [24] Section 5.9), which will be used many times in the various derivations of this chapter, and which can be easily derived from the Taylor series expansion of a function $f(x)$ about the point $x = a$. The expansion is given by

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + f'''(a)\frac{(x-a)^3}{3!} + \dots,$$

where $f'(a)$ means the derivative of $f(x)$ with respect to x evaluated at the point $x = a$, $f''(a)$ is the second derivative of $f(x)$ with respect to x evaluated at the point $x = a$, etc. If, in the above equation we let

$$x = a + h \quad \text{and} \quad a = x,$$

we obtain

$$f(x+h) = f(x) + f'(x)h + f''(x)\frac{h^2}{2!} + f'''(x)\frac{h^3}{3!} + \dots$$

On the assumption that h is small in absolute value compared with unity, the terms of the series involving high powers of h may be neglected, and we may write as a first approximation

$$f(x+h) = f(x) + f'(x)h. \quad (1)$$

Thus the value of $f(x)$ at the point $x+h$ (near to x) is given by the value of the function at the point x plus the value of its first derivative *evaluated at x* , multiplied by the distance $[(x+h) - x = h]$ between the two points.

A typical example of the use of Equation (1) is the following. Suppose the value of the partial derivative of a function $u(x,y)$ with respect to x at the point x , that is $\partial u/\partial x$, is known. Then the value of $\partial u/\partial x$ at the point $x+dx$ is given by Equation (1) as

$$\left(\frac{\partial u}{\partial x}\right)_{x+dx} = \left(\frac{\partial u}{\partial x}\right)_x + \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x}\right)_x (x+dx - x),$$

or

$$\left(\frac{\partial u}{\partial x}\right)_{x+dx} = \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} dx.$$

The subscripts x and $x + dx$ indicate that the value of the derivative of $u(x, y)$ with respect to x must be evaluated at the points (x, y) and $(x + dx, y)$, respectively. However, the subscript x is generally omitted, our use of it in the above example being merely for emphasis.

2. The vibrating string

As a first example of a partial differential equation we shall derive the equation satisfied by the small transverse displacements of a vibrating string. The sketch in Figure 1 represents a tightly

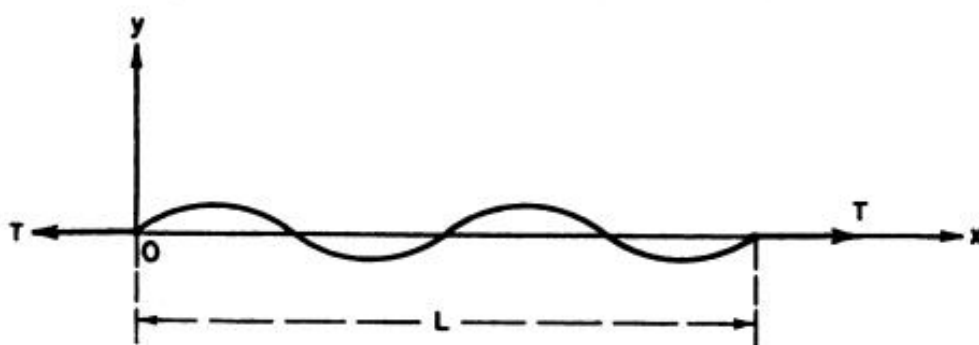


Fig. 1

stretched string of length L , fixed at the end points, and subjected to a constant tension of T pounds. Our problem is to determine what equation governs the motion of the string under various initial conditions and to find a function $u(x, t)$ which gives the deflection of the string at any point x and at any time t .

In order to obtain a simple differential equation we shall consider a more or less "idealized" string. Presumably, any physical string will reasonably well approximate the "ideal string," and hence the function $u(x, t)$ obtained from the simplified equation will reasonably well represent the actual displacement. We make the following simplifying assumptions.

1. The string is perfectly flexible, that is, it cannot resist bending moments.

2. The deflections u are small compared with the length L of the string.

3. The slope at any point of the deflected string is small compared with unity.

4. The tension T is constant at all times and at all points of the deflected string, and is large compared with the weight of the string.

5. The horizontal displacement of the particles of the string is negligible compared with the vertical displacement, that is, we have pure transverse vibrations.

6. The motion takes place only in the xy -plane.

If we check through the above six assumptions we see that they are all reasonable and *consistent*. By this we mean that no contradictions will arise from the acceptance of the truth of all the above assumptions. (An example of a pair of inconsistent assumptions would be to assume that the displacements u are large compared with the length of the string, but that the horizontal displacements are small.)

Consider a small differential segment ds of the string (see Figure 2), and let w be the weight per unit length of the string. From elementary mechanics, $\Sigma F = ma$; in words, the sum of the

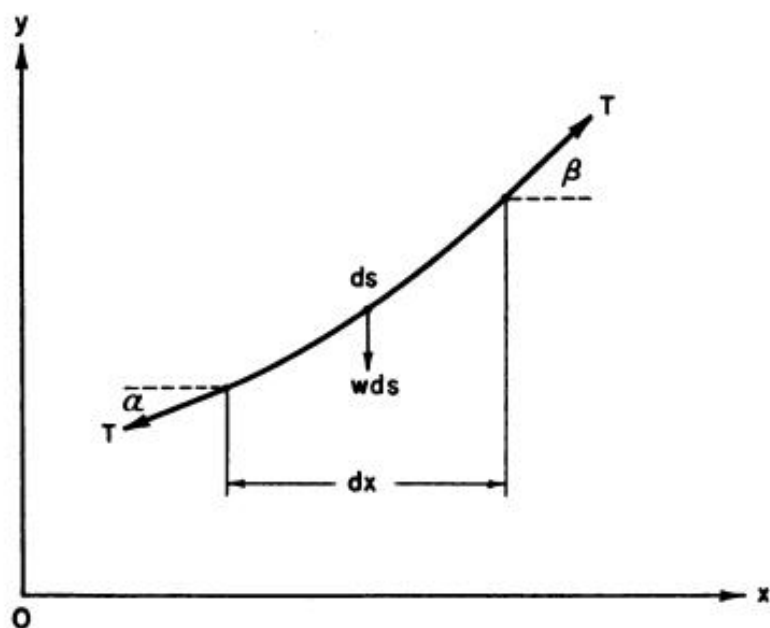


Fig. 2

forces in any direction applied to the string segment is equal to the product of the mass of the segment by the acceleration in that direction. In particular, the sum of the forces in the y direction is

$$\Sigma F_y = -T \sin \alpha + T \sin \beta - w ds.$$

Since the deflections are small, we may replace $\sin \alpha$ and $\sin \beta$ by $\tan \alpha$ and $\tan \beta$, respectively. Also, by the third assumption $dy \ll dx$ and $ds = dx$ to first-order approximation. Hence the sum of the forces in the y -direction may be written in the form

$$\Sigma F_y = -T \tan \alpha + T \tan \beta - w dx.$$

The slope at any point x of the deflected string is, from the calculus,

$$\tan \alpha = \frac{\partial u}{\partial x},$$

and by Equation (1), $\tan \beta$ is given to first-order approximation by

$$\tan \beta = \frac{\partial u}{\partial x} + \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) dx = \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} dx.$$

Hence ΣF_y becomes

$$\begin{aligned} \Sigma F_y &= -T \frac{\partial u}{\partial x} + T \frac{\partial u}{\partial x} + T \frac{\partial^2 u}{\partial x^2} dx - w dx \\ &= T \frac{\partial^2 u}{\partial x^2} dx - w dx. \end{aligned} \quad (2)$$

This sum of forces in the y -direction must be equated to the product of the mass m and acceleration a_y in the y -direction. The acceleration a_y in the y -direction is the second derivative of distance with respect to time, namely,

$$a_y = \frac{\partial^2 u}{\partial t^2},$$

and hence

$$ma_y = \frac{w}{g} dx \frac{\partial^2 u}{\partial t^2}, \quad (3)$$

where g is the gravitational constant. By Newton's second law we obtain, from Equations (2) and (3),

$$T \frac{\partial^2 u}{\partial x^2} dx - w dx = \frac{w}{g} dx \frac{\partial^2 u}{\partial t^2},$$

or, dividing by dx and multiplying by g/w ,

$$\frac{gT}{w} \frac{\partial^2 u}{\partial x^2} - g = \frac{\partial^2 u}{\partial t^2}. \quad (4)$$

The constant gT/w has the dimensions of velocity squared, and it is customary to write

$$\frac{gT}{w} = a^2,$$

where a has the dimensions of velocity. Also, by the fourth assumption, g may be neglected in comparison with $\partial^2 u / \partial t^2$, and Equation (4) assumes the standard form

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}. \quad (5)$$

This is the partial differential equation satisfied by the vertical motion $u(x, t)$ of the vibrating string. It is also called the one-dimensional *wave equation*.

3. Boundary conditions

In solving a physical problem by means of differential equations we are frequently interested in the solution of the differential equation which satisfies a given set of initial or *boundary* conditions. The boundary conditions for the case of the vibrating string are readily determined by inspection as

1. $u(0, t) = 0$,
 2. $u(L, t) = 0$,
 3. $\frac{\partial}{\partial t} u(x, 0) = g(x),^*$
 4. $u(x, 0) = f(x).$
- (6)

* $\frac{\partial}{\partial t} u(x, 0)$ means the derivative of $u(x, t)$ with respect to t evaluated at $t = 0$.

This notation is consistent with that used for ordinary derivatives (see Section 1) where we write $f'(a)$ to indicate the derivative of $f(x)$ with respect to x evaluated at $x = a$. Other more or less standard notations for $\frac{\partial}{\partial t} u(x, 0)$ are

$$\left. \frac{\partial}{\partial t} u(x, t) \right|_{t=0}, \quad u_t(x, t) \Big|_{t=0}, \quad u_t(x, 0).$$

The first two boundary conditions state that the string is fixed at the end points, $x = 0$ and $x = L$ for all time t . The third states that initially the transverse velocity of the string is a given function of x , namely $g(x)$, while the fourth indicates that the original shape of the displaced string (that is at time $t = 0$) is another specified function $f(x)$. [If the first two boundary conditions are to be satisfied, $f(0) = f(L) = 0$.]

A question arising in connection with Equation (6) is whether this is the requisite number of boundary conditions, that is, whether one or many or perhaps no functions exist which satisfy the differential equation and these four boundary conditions. By physical reasoning it seems reasonable that in this example, at least, there exists a unique solution, but in general no conclusive answer can be given to the question of whether the boundary conditions determine a unique solution, many solutions, or no solution at all. Certain remarks, however, may be made in this connection. First, we cannot get new boundary conditions by differentiating known boundary conditions. For example,

$$\frac{\partial}{\partial x} u(x, 0) = f'(x) \quad \text{or} \quad \frac{\partial^2}{\partial x \partial t} u(x, 0) = g'(x)$$

are not additional boundary conditions. Second, if the initial position $f(x)$ and velocity $g(x)$ of the particles of the string have been specified, the acceleration, say

$$\frac{\partial^2}{\partial t^2} u(x, 0) = \psi(x),$$

and higher time derivatives cannot be specified. For, once we have specified $u(x, 0)$ and

$$\frac{\partial}{\partial t} u(x, 0),$$

the higher time derivatives are uniquely *determined* by the differential equation. This situation is readily illustrated in the case of the elementary ordinary differential equation $\ddot{x} + kx = 0$. Here, once the initial position and velocity, say $\dot{x}(0) = a$, $\ddot{x}(0) = b$ are specified, we obtain $\ddot{x}(0) = -ka$, $\ddot{\ddot{x}}(0) = -kb$, etc. from the differential equation itself. If we were to specify $x(0)$, $\dot{x}(0)$, and

$\ddot{x}(0)$ independently, we would have no solution to the problem, unless $\ddot{x}(0)$ were fortuitously chosen as $-kx(0)$.

A differential equation together with its boundary conditions is said to form a *differential system*, and the modern theory of partial differential equations is concerned with differential systems and not with differential equations alone. In the classical theory of ordinary differential equations it is frequently not too difficult a task to find the general solution of the equation involving various arbitrary constants and then determine these constants so as to satisfy the boundary conditions. The equivalent problem in partial differential equations is much more difficult, and we must look at each differential system as a new problem even though the differential *equation* part of different systems may be identical. Considering differential systems rather than differential equations alone is not too great a restriction, for if the solution is to have physical meaning it must satisfy *some* boundary conditions.

To summarize then, our problems will be concerned with differential systems, or as we shall say, with *boundary value problems*, and we shall solve particular partial differential equations subject to certain specific boundary conditions.

4. The vibrating membrane

As a second example we shall derive the equation satisfied by the transverse motion of a vibrating drumhead or membrane. Such a two-dimensional membrane is illustrated in Figure 3.* Our problem is to find the equation satisfied by the deflection $u(x, y, t)$ of a thin membrane oriented in the xy -plane. As in the case of the vibrating string certain "idealizing" assumptions will be made.

1. The force per unit length, S , is constant and everywhere normal to the boundary of the membrane.
2. The total force on the boundary is large compared with the weight of the membrane.

* The cross in the circle indicates that the x -axis is *into* the paper, while a dot in the circle (see, for example, Figure 4) indicates that the axis is *out* of the plane of the paper.

3. The membrane is so thin that it cannot resist bending moments.

4. The deflections u are small compared with the lateral dimensions of the membrane.

5. The slopes are small compared with unity.

6. The lateral displacements are negligible compared with the vertical displacements.

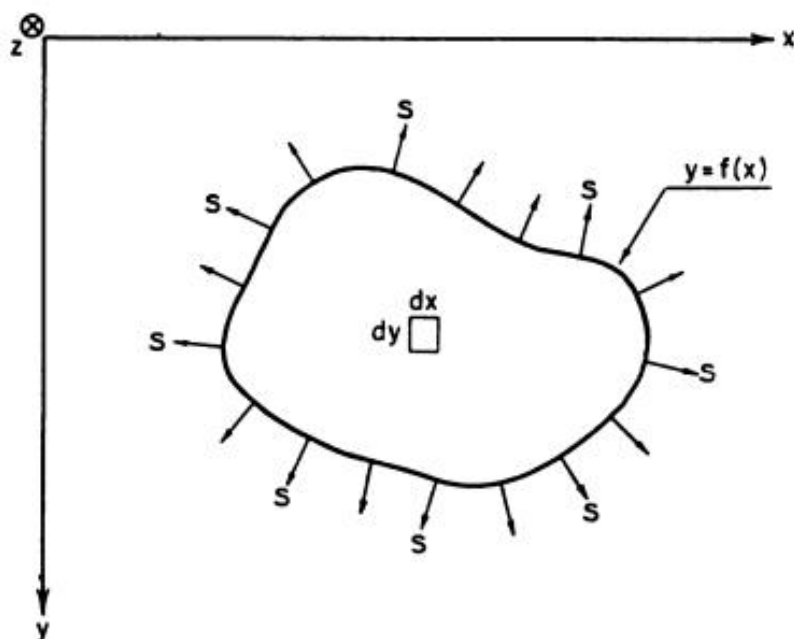


Fig. 3

Consider a differential element of the membrane of area $dx dy$ as indicated in Figure 3. If at a fixed time t a plane parallel to the xz -plane is passed through this element, the cross section of the element $dx dy$ will appear as in Figure 4. Similarly a plane passed parallel to the yz -plane will yield a section as in Figure 5. If w is the weight per unit area of the membrane and S the constant normal force per unit length, the sum of the forces (ΣF_z) in the z -direction is given by

$$\Sigma F_z = -S dy \tan \alpha + S dy \tan \beta - S dx \tan \gamma + S dx \tan \delta + w dx dy, \quad (7)$$

since by virtue of the fourth assumption, slopes are small and $\sin \alpha$, $\sin \beta$, $\sin \gamma$, $\sin \delta$ are approximately equal to $\tan \alpha$, $\tan \beta$, $\tan \gamma$, and $\tan \delta$, respectively.

From the calculus,

$$\tan \alpha = \frac{\partial u}{\partial x} \quad \text{and} \quad \tan \gamma = \frac{\partial u}{\partial y},$$

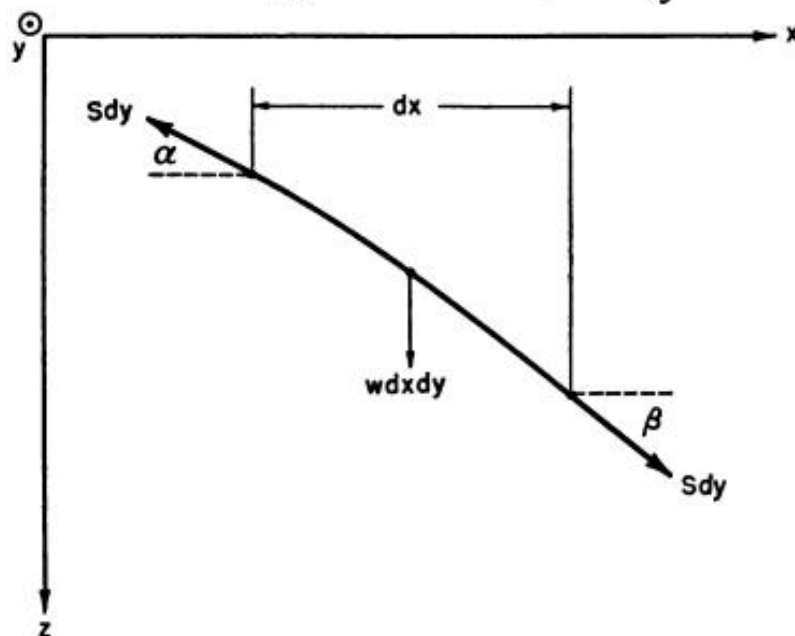


Fig. 4

and from Equation (1),

$$\tan \beta = \frac{\partial u}{\partial x} + \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) dx,$$

$$\tan \delta = \frac{\partial u}{\partial y} + \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) dy.$$

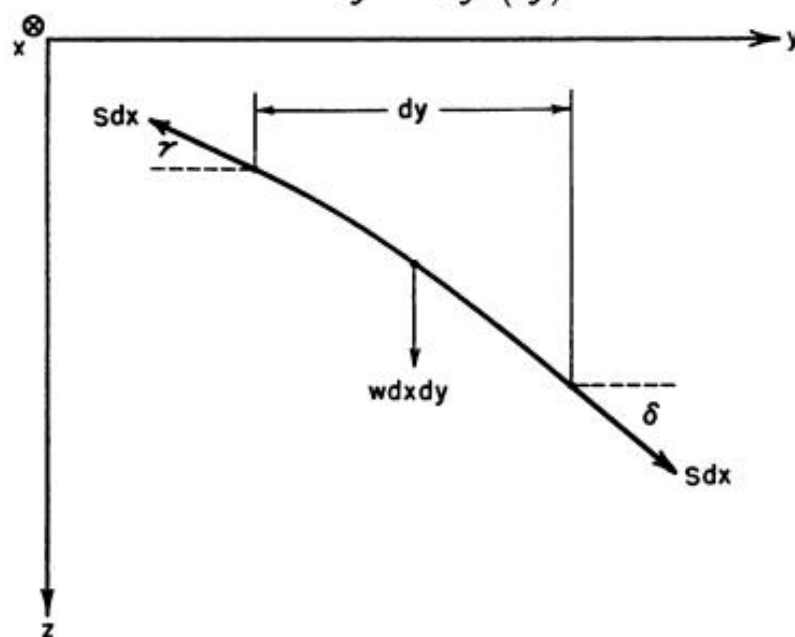


Fig. 5

Using these results, Equation (7) becomes

$$\begin{aligned}\Sigma F_z = & -S dy \frac{\partial u}{\partial x} + S dy \left(\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} dx \right) \\ & - S dx \frac{\partial u}{\partial y} + S dx \left(\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y^2} dy \right) + w dx dy.\end{aligned}$$

Simplifying, and noting that $w dx dy$ is negligible by the second assumption, ΣF_z assumes the form

$$\Sigma F_z = S dx dy \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

From the fundamental law of mechanics, $\Sigma F_z = ma_z$, where a_z is the acceleration in the z -direction. The mass of the elementary rectangle $dx dy$ is

$$m = \frac{w}{g} dx dy,$$

and its acceleration is

$$a_z = \frac{\partial^2 u}{\partial t^2}.$$

Hence

$$S dx dy \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{w}{g} dx dy \frac{\partial^2 u}{\partial t^2}.$$

Dividing by $dx dy$ and multiplying through by g/w , this equation becomes

$$\frac{gS}{w} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial^2 u}{\partial t^2}.$$

As in the case of the vibrating string, gS/w has the dimensions of velocity squared, $gS/w = a^2$, where a has the dimensions of velocity. The equation of the vibrating membrane then assumes the classical form

$$a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = \frac{\partial^2 u}{\partial t^2}, \quad (8)$$

and is often called the wave equation in two dimensions.

In order to determine a differential system we must associate boundary conditions with Equation (8). Typical boundary conditions might give the initial displacement and velocity of the membrane and state that the boundary is held fixed for all time.

These conditions can be expressed mathematically as

1. $u(x, f(x), t) = 0$
 2. $\frac{\partial}{\partial t} u(x, y, 0) = \psi(x, y)$
 3. $u(x, y, 0) = \phi(x, y)$
- (9)

where $y = f(x)$ is the equation (in the xy -plane) of the curve bounding the membrane (see Figure 3). Condition one states that the boundary of the membrane is held fixed and is analogous to the first two boundary conditions of the vibrating string problem. The second boundary condition asserts that the initial velocity of the membrane is a prescribed function of x and y , while the third boundary condition gives the initial displacement of the membrane. Note that $\phi(x, y)$ is not quite arbitrary, for if the first boundary condition is to be satisfied, $\phi(x, y)$ must vanish on the boundary.

Our results can be extended to the case of a three-dimensional vibrating elastic body. In this case the differential system assumes the form

$$a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = \frac{\partial^2 u}{\partial t^2} \quad (10)$$

with appropriate boundary conditions (see [17]). Equation (10) is also called the wave equation in three dimensions.

5. Longitudinal vibrations of a bar

Another example of a physical phenomenon giving rise to a partial differential equation is that of the longitudinal vibrations of a bar or of a long helical spring. The simplifying assumptions usually made regarding the bar are :

1. The material is homogeneous, isotropic, and elastic; plane sections remain plane.
2. The material satisfies Hooke's law : $\sigma = E\epsilon$, where σ is the stress (psi), E is Young's modulus (psi), and ϵ is the unit strain (inches/inch).
3. The displacements are small in comparison with the dimensions of the bar.

Referring to Figure 6, we let

dx = the length of a differential element of the bar,

u = displacement of a given section of the bar at a given time t ,

ϵ = unit strain = $\frac{(\text{new length}) - (\text{old length})}{\text{old length}}$.

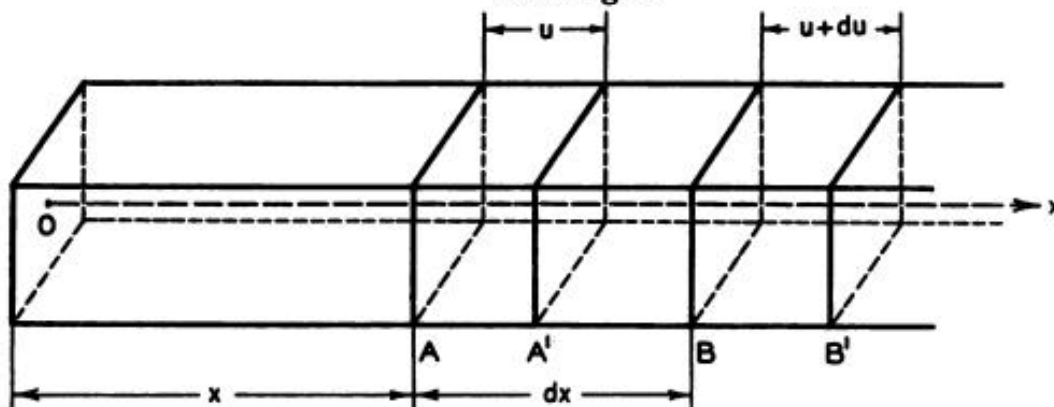


Fig. 6

Now,

new length = $A'B'$

$$= [(x + dx) + (u + du)] - (x + u) = dx + du,$$

old length = $AB = dx$.

Hence

$$\epsilon = \frac{(dx + du) - dx}{dx} = \frac{\partial u}{\partial x}.$$

Let $A(x)$ be the cross-sectional area of the bar at x . This is shown to be a rectangle in Figure 6, but could be any shape. Let F_x be the total force in pounds at the point x , assumed positive when tensile (see Figure 7). The force F_x is equal to the stress σ multiplied by the cross-sectional area,

$$F_x = \sigma A.$$

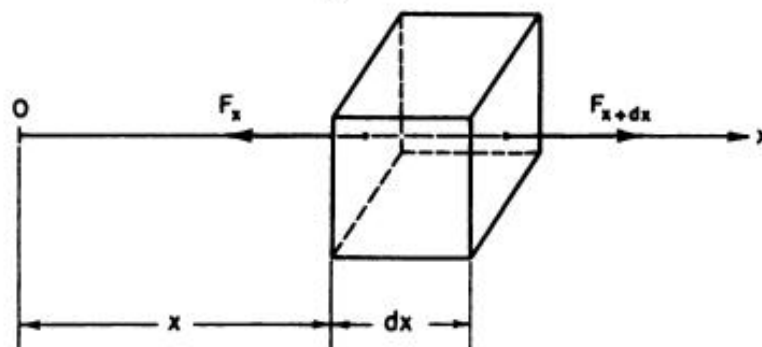


Fig. 7

From our second assumption $\sigma = E\epsilon$, and with $\epsilon = \partial u / \partial x$,

$$F_x = \sigma A = E\epsilon A = EA \frac{\partial u}{\partial x}. \quad (11)$$

Hence the force F_{x+dx} at the point $x + dx$ is, by Equation (1),

$$F_{x+dx} = EA \frac{\partial u}{\partial x} + \frac{\partial}{\partial x} \left(EA \frac{\partial u}{\partial x} \right) dx. \quad (12)$$

Therefore by the fundamental law of dynamics, namely the sum of the forces in a given direction is equal to the mass times the acceleration in that direction, we obtain

$$F_{x+dx} - F_x = \left(\frac{w}{g} A dx \right) \frac{\partial^2 u}{\partial t^2}, \quad (13)$$

where w is the weight per unit volume of the bar and consequently $(w/g)A dx$ is the mass of the element shown in Figure 7; and $\partial^2 u / \partial t^2$ is the acceleration. From Equations (11) and (12),

$$F_{x+dx} - F_x = \frac{\partial}{\partial x} \left(EA \frac{\partial u}{\partial x} \right) dx,$$

and from Equation (13),

$$\frac{\partial}{\partial x} \left(EA \frac{\partial u}{\partial x} \right) dx = \frac{w}{g} A \frac{\partial^2 u}{\partial t^2} dx.$$

Dividing by dx and multiplying by g/wA , the equation assumes the form

$$\frac{g}{wA} \frac{\partial}{\partial x} \left(EA \frac{\partial u}{\partial x} \right) = \frac{\partial^2 u}{\partial t^2}, \quad (14)$$

which is the equation governing the longitudinal vibrations of the bar. If we make the additional assumption that EA is constant,

$$\frac{\partial}{\partial x} \left(EA \frac{\partial u}{\partial x} \right) = EA \frac{\partial^2 u}{\partial x^2},$$

and Equation (14) takes the simple form

$$\frac{gE}{w} \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}.$$

The constant $a^2 = gE/w$ has the dimensions of velocity squared, and by its use Equation (14) now assumes the classical form

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad (15)$$

which we recognize as the one-dimensional wave equation. We see that Equation (15) is identical with that of the vibrating string derived in Section 2. Here, then, is our first example of two different physical phenomena giving rise to the same differential equation.

6. Transverse vibrations of a beam

A beam of length L loaded by a variable load of $w(x,t)$ lb/ft is sketched in Figure 8 and referred to a y -axis positive downward and an x -axis positive to the right. We wish to find the *transverse* vibrations $y(x,t)$ of this beam due to a set of initial conditions.

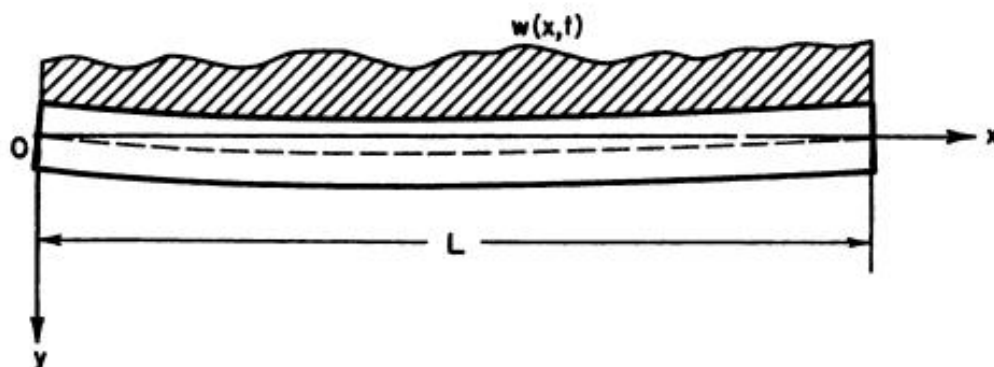


Fig. 8

The weight of the beam itself will be assumed to be included in the load $w(x,t)$. The "idealizing" assumptions made to set up this equation are those of simple beam theory (that is, the assumptions of the classical Bernoulli-Euler theory), from which it may be shown that (see [25] Section 2.10)

$$M = -EIC, \quad (16)$$

where M is the bending moment (ft-lb), E is Young's modulus (psf), I is the moment of inertia of a transverse section of the beam (ft⁴), and C is the curvature of the beam (ft⁻¹). Since from the calculus ([24], Section 2.7), the curvature (reciprocal of the radius of curvature) is given in Cartesian coordinates by

$$C = \frac{y''}{[1 + y'^2]^{3/2}},$$

and by the small slope assumption of elementary beam theory,

$$y' \ll 1,$$

we have to first-order approximation,

$$C = y''.$$

Equation (16) then assumes the form

$$M = -EI \frac{\partial^2 y}{\partial x^2}. \quad (17)$$

The quantity EI is known as the "flexural rigidity" of the beam.

The equations of motion of an element dx of the beam are given by Newton's second law :

$$\Sigma F_y = ma_y,$$

$$\Sigma M = J\alpha.$$

The first states that the sum of the forces in the y -direction is equal to the mass times the acceleration in the same direction, while the second states that the sum of the external moments about any point is equal to the moment of inertia of the element times its angular acceleration. Since we are considering pure transverse motion, the angular acceleration is assumed to be zero, and the above equations reduce to

$$\Sigma F_y = ma_y, \quad \Sigma M = 0.$$

Let us apply these equations to the element dx illustrated in Figure 9, in which, with the coordinate axes of Figure 8, shears Q

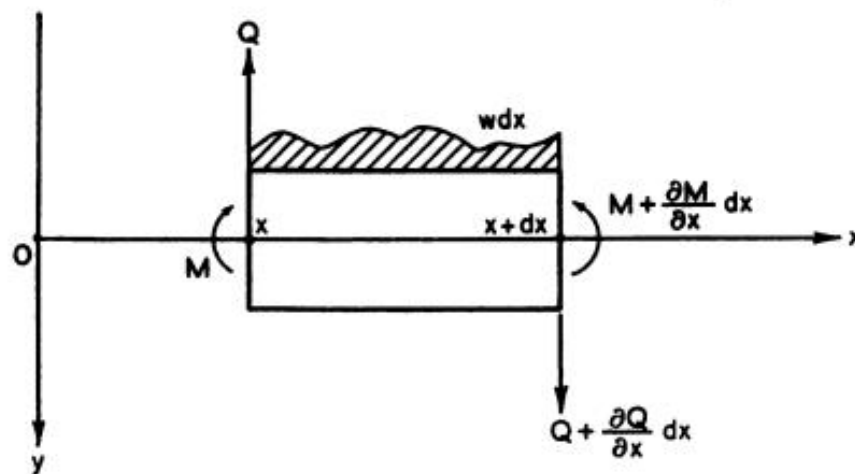


Fig. 9

and moments M are positive as indicated. The sum of the forces in the y -direction, ΣF_y , is

$$\Sigma F_y = -Q + \left(Q + \frac{\partial Q}{\partial x} dx \right) + w dx = \frac{\partial Q}{\partial x} dx + w dx.$$

The mass of the element is $(w/g) dx$ and the acceleration in the y -direction is $\partial^2 y / \partial t^2$. Hence

$$\frac{\partial Q}{\partial x} dx + w dx = \frac{w}{g} \frac{\partial^2 y}{\partial t^2} dx$$

or, dividing by dx ,

$$\frac{\partial Q}{\partial x} + w = \frac{w}{g} \frac{\partial^2 y}{\partial t^2}. \quad (18)$$

In order to eliminate the shear Q and obtain an equation involving only y and its derivatives, we utilize the moment equation $\Sigma M = 0$. Taking moments about the point x of Figure 9,

$$\begin{aligned} \Sigma M &= +M - \left(M + \frac{\partial M}{\partial x} dx \right) + \left(Q + \frac{\partial Q}{\partial x} dx \right) dx + (w dx) \frac{dx}{2} \\ &= -\frac{\partial M}{\partial x} dx + Q dx + \frac{\partial Q}{\partial x} dx^2 + w \frac{dx^2}{2}. \end{aligned}$$

But terms involving dx^2 are infinitesimals of higher order ([24], Section 1.6) with respect to terms involving dx , and hence can be neglected. Therefore

$$\Sigma M = -\frac{\partial M}{\partial x} dx + Q dx = 0, \quad (19)$$

or canceling dx ,

$$-\frac{\partial M}{\partial x} + Q = 0. \quad (20)$$

Differentiating Equation (20) with respect to x ,

$$\frac{\partial Q}{\partial x} = \frac{\partial^2 M}{\partial x^2},$$

and substituting this expression in Equation (18), we obtain an equation involving only y and the moment M , namely,

$$\frac{\partial^2 M}{\partial x^2} + w = \frac{w}{g} \frac{\partial^2 y}{\partial t^2}. \quad (21)$$

Finally, differentiating Equation (17) twice with respect to x ,

$$\frac{\partial^2 M}{\partial x^2} = -\frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 y}{\partial x^2} \right),$$

and substituting in Equation (21), we obtain an equation involving only y :

$$-\frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 y}{\partial x^2} \right) + w = \frac{w}{g} \frac{\partial^2 y}{\partial t^2}.$$

This is the partial differential equation satisfied by the transverse motions of a beam. If the load $w(x,t)$ can be neglected with respect to the inertia forces $(w/g)(\partial^2 y/\partial t^2)$, the above equation assumes the form

$$\frac{w}{g} \frac{\partial^2 y}{\partial t^2} + \frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 y}{\partial x^2} \right) = 0.$$

If, further, the flexural rigidity EI is constant, we obtain the additional simplification,

$$\frac{w}{g} \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} = 0,$$

or, on letting $a^2 = EIg/w$,

$$\frac{\partial^2 y}{\partial t^2} + a^2 \frac{\partial^4 y}{\partial x^4} = 0. \quad (22)$$

The quantity a^2 is not a constant unless $w(x,t)$ is constant. We leave as an exercise for the reader the problem of determining the dimensions of a and of verifying the dimensional correctness of Equation (22).

Common sets of boundary conditions associated with Equation (22) state that the beam is "simply supported" (Figure 10)

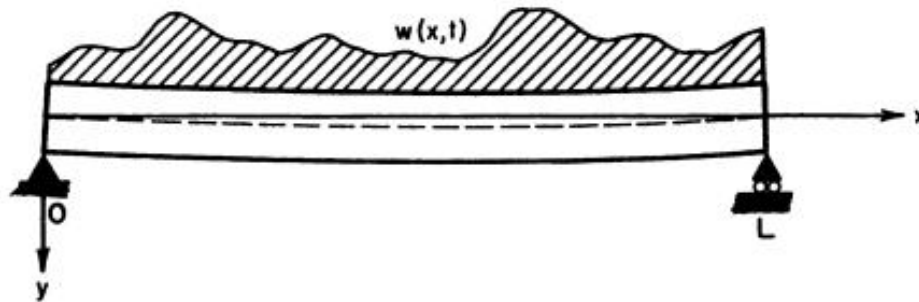


Fig. 10

or "built-in" at both ends (Figure 11). When the beam is "simply supported" these conditions are

1. $y(0,t) = 0,$
 2. $y(L,t) = 0,$
 3. $\frac{\partial^2}{\partial x^2} y(0,t) = 0,$
 4. $\frac{\partial^2}{\partial x^2} y(L,t) = 0,$
 5. $y(x,0) = f(x),$
 6. $\frac{\partial}{\partial t} y(x,0) = g(x).$
- (23)

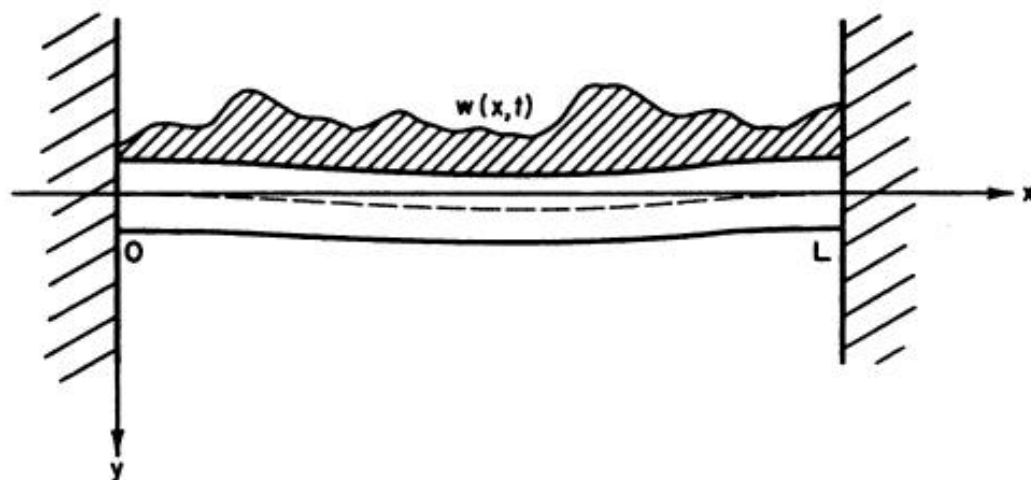


Fig. 11

The first two boundary conditions state that the end points 0 and L are fixed for all time t . The third and fourth conditions result from the fact that a simply supported beam has zero moment and hence zero curvature at its free ends. The fifth gives the initial displacement of the beam and the sixth the initial velocity.

The boundary conditions for the built-in beam are

1. $y(0,t) = 0,$
2. $y(L,t) = 0,$
3. $\frac{\partial}{\partial x} y(0,t) = 0,$

$$\begin{aligned}
 4. \quad & \frac{\partial}{\partial x} y(L, t) = 0, \\
 5. \quad & y(x, 0) = f(x), \\
 6. \quad & \frac{\partial}{\partial t} y(x, 0) = g(x).
 \end{aligned}
 \tag{24}$$

As before, the first two conditions state that the end points 0 and L are fixed, while the third and fourth result from the fact that the slope of the deflection of a built-in beam is zero at the end points. The fifth and sixth give the initial transverse displacement and velocity of each particle, respectively.

7. Heat flow in one dimension

The problems that appear in the next few sections will involve the flow of heat. These equations are of interest to the chemical engineer and physicist as well as to the applied mathematician. Our first application will be to the flow of heat in one dimension.

Suppose we have a long thin bar or wire oriented along the x -axis, and we wish to determine the temperature distribution $u(x, t)$ in the bar. We shall assume that the bar is perfectly insulated laterally, and that heat flows in one dimension only (the x -direction). The following physical laws will be used :

1. The heat contained in an element of mass m is proportional to the mass and to the absolute temperature, that is,

$$Q = cmu, \tag{25}$$

where, Q = heat (calories or Btu),
 c = specific heat,
 m = mass,
 u = absolute temperature.

2. The rate of heat flow through a surface is proportional to the area of the surface and to the outward normal gradient of temperature, that is,

$$\frac{\partial Q}{\partial t} = -kA \frac{\partial u}{\partial n} \text{ cal/sec or Btu/sec,} \tag{26}$$

where, A = cross-sectional area of surface in a direction perpendicular to the flow of heat,
 $\partial u / \partial n$ = gradient of temperature perpendicular to the surface,
 k = thermal conductivity.

The negative sign in Equation (26) follows from the fact that heat flows *into* the body if the temperature to the right of the boundary is lower than the temperature to the left, that is, if $\partial u / \partial n$ is negative.

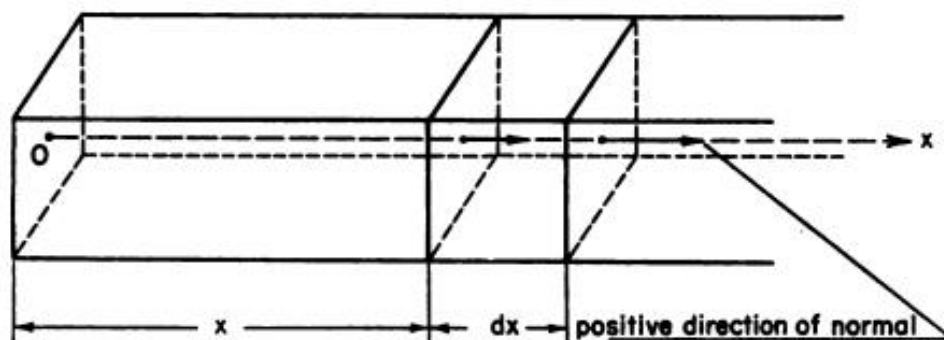


Fig. 12

Figure 12 represents the insulated bar. If we consider a small segment dx of the bar, the amount of heat Q_1 in this elementary element is, from Equation (25),

$$Q_1 = c(A dx) \delta u,$$

where δ is the density (mass per unit volume) of the material of the bar, and hence $A dx \delta$ is its mass. The *rate* at which heat is stored in the element of volume $A dx$ is therefore $\partial Q_1 / \partial t$,

$$\frac{\partial Q_1}{\partial t} = cA\delta dx \frac{\partial u}{\partial t}. \quad (27)$$

If we assume that the positive direction of heat flow is the x -axis we find from Equation (26) that the rate at which heat flows *into* the element is

$$\frac{\partial Q_2}{\partial t} = -kA \frac{\partial u}{\partial x}, \quad (28)$$

and the rate at which heat is flowing *out* of the element is, by Equation (1),

$$\frac{\partial Q_3}{\partial t} = \frac{\partial Q_2}{\partial t} + \frac{\partial}{\partial x} \left(\frac{\partial Q_2}{\partial t} \right) dx = -kA \frac{\partial u}{\partial x} - \frac{\partial}{\partial x} \left(kA \frac{\partial u}{\partial x} \right) dx. \quad (29)$$

The rate of heat flowing *in*, minus the rate of heat flowing *out*, is the rate at which heat is stored in the elementary volume $A \, dx$. Hence

$$\frac{\partial Q_1}{\partial t} = \frac{\partial Q_2}{\partial t} - \frac{\partial Q_3}{\partial t},$$

or, by Equations (27), (28), and (29),

$$cA\delta \frac{\partial u}{\partial t} dx = \frac{\partial}{\partial x} \left(kA \frac{\partial u}{\partial x} \right) dx.$$

Dividing by dx , we obtain the equation

$$cA\delta \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(kA \frac{\partial u}{\partial x} \right).$$

If we assume k and A to be independent of x , we may also write

$$\frac{\partial^2 u}{\partial x^2} = \frac{c\delta}{k} \frac{\partial u}{\partial t}, \quad (30)$$

which is the standard equation for the flow of heat in one dimension.

8. The three-dimensional heat equation

The above results are readily generalized to the flow of heat in a three-dimensional body. Figure 13 represents an elementary

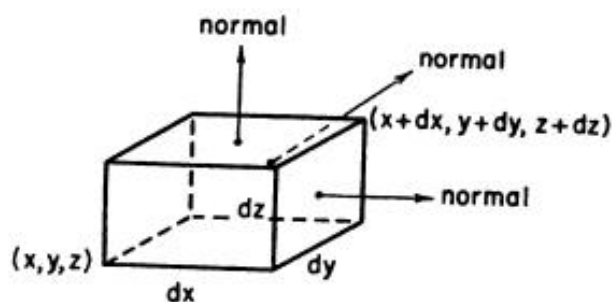


Fig. 13

parallelepiped $dx \, dy \, dz$ in the three-dimensional body. If we let $u(x, y, z, t)$ be the temperature at any point x, y, z and at any time t in the volume, then by the fundamental law of Equation (25),

$$\frac{\partial Q_1}{\partial t} = c(dx \, dy \, dz) \delta \frac{\partial u}{\partial t} \quad (31)$$

is the rate of heat stored in the elementary volume, where δ is the density of the material. From Equation (26), the rate at which heat flows *into* the solid through the three faces at x, y, z , normal to the coordinate axes, is given by

$$\frac{\partial Q_2}{\partial t} = -k \, dx \, dy \, \frac{\partial u}{\partial x} - k \, dy \, dz \, \frac{\partial u}{\partial y} - k \, dz \, dx \, \frac{\partial u}{\partial z}, \quad (32)$$

and the rate at which heat flows *out* through the three faces at $x + dx, y + dy, z + dz$ is given by

$$\begin{aligned} \frac{\partial Q_3}{\partial t} = & -k \, dx \, dy \left(\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} \, dx \right) - k \, dy \, dz \left(\frac{\partial u}{\partial y} + \frac{\partial^2 u}{\partial y^2} \, dy \right) \\ & - k \, dz \, dx \left(\frac{\partial u}{\partial z} + \frac{\partial^2 u}{\partial z^2} \, dz \right), \end{aligned} \quad (33)$$

where we have assumed that the thermal conductivity k is independent of x, y , and z (and hence can be taken outside the differentiation signs). As in Section 7, the rate of heat that flows into the body, $\partial Q_2/\partial t$, less the rate of heat that flows out of the elementary volume, $\partial Q_3/\partial t$, is the rate of accumulation of heat $\partial Q_1/\partial t$ in the body. Expressing this fact in formulas gives

$$\frac{\partial Q_1}{\partial t} = \frac{\partial Q_2}{\partial t} - \frac{\partial Q_3}{\partial t}.$$

Substituting Equations (31), (32), and (33) in the above equation,

$$c\delta \frac{\partial u}{\partial t} \, dx \, dy \, dz = k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) dx \, dy \, dz$$

or

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{c\delta}{k} \frac{\partial u}{\partial t}, \quad (34)$$

which is the standard form of the heat equation in three dimensions.

The differential operator

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

which was met before in the case of the vibrating elastic solid, is called the three-dimensional *Laplacian*, and is symbolized by ∇^2 .

Using this standard notation, the heat flow equation may be written as

$$\nabla^2 u = \frac{c\delta}{k} \frac{\partial u}{\partial t}.$$

If the temperature u is independent of time, $\partial u/\partial t$ is zero, and

$$\nabla^2 u = 0.$$

For the case of heat flow in two dimensions, for example, in a thin flat slab, insulated on both faces where we assume the flow takes place only in the xy -directions, the equation assumes the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{c\delta}{k} \frac{\partial u}{\partial t}. \quad (35)$$

This equation is also written

$$\nabla^2 u = \frac{c\delta}{k} \frac{\partial u}{\partial t},$$

where here ∇^2 is the two-dimensional Laplacian. Whether ∇^2 represents

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad \text{or} \quad \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

will be clear from the context.

9. The transmission line equations

As an example of partial differential equations which arise in electrical engineering we shall derive the *transmission line* or *telegraph equations*. Suppose we have a transmission line which we shall consider a distributed constant-parameter system. By this we mean that the resistance, inductance, and capacitance vary linearly with the space coordinate x , that is, any small section dx of the line will contain the same amount of resistance, inductance, and capacitance as any other element of length dx , (in contradistinction to a *lumped parameter system* in which we assume that we have a finite number of resistors, coils, and capacitors connected by impedanceless wires). In a distributed parameter system the voltage and current will vary along the length of the

line, and in general, will also vary as a function of time. The problem is to determine the equations satisfied by the current, $i(x,t)$, and voltage, $e(x,t)$, distributions in the line. Referring to Figure 14, let

R = series resistance in ohms per loop mile,

L = series inductance in henrys per loop mile,

C = shunt capacitance in farads per mile,

G = shunt conductance in mhos per mile.

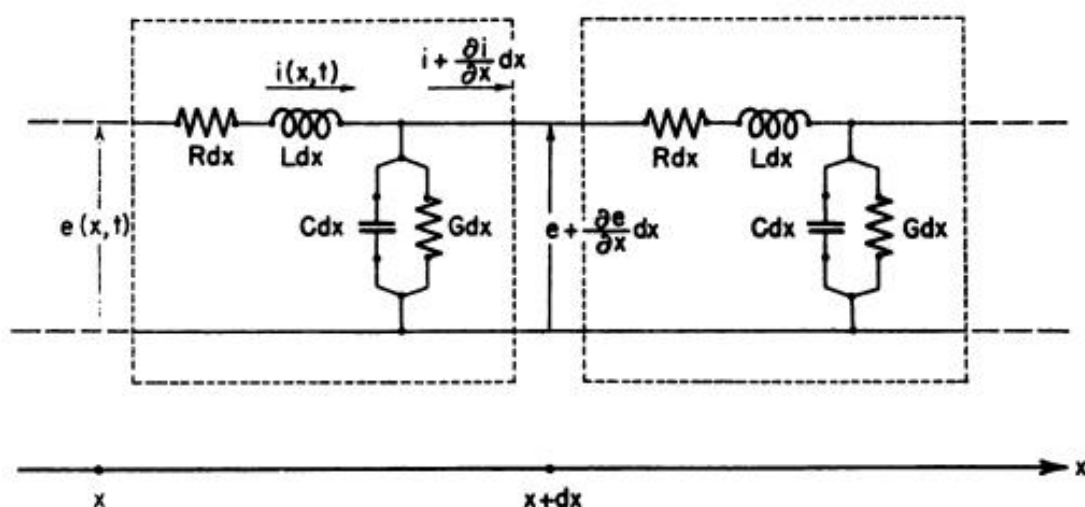


Fig. 14

According to Kirchhoff's first law, the drop in potential around a closed loop equals the rise in potential. Considering an elementary section dx , the rise in potential at the point x is $e(x,t)$ while the rise in potential at the point $x + dx$ is

$$e(x,t) + \frac{\partial e}{\partial x} dx.$$

Hence the net rise around the loop in the clockwise direction is

$$e(x,t) - \left[e(x,t) + \frac{\partial e}{\partial x} dx \right] = -\frac{\partial e}{\partial x} dx.$$

The drop in potential around the loop is equal to the sum of the resistive drops and inductive drops. The drop in the resistance $R dx$ is

$$iR dx,$$

and the drop in potential due to the inductance $L dx$ is

$$L \frac{\partial i}{\partial t} dx.$$

Hence

$$-\frac{\partial e}{\partial x} dx = iR dx + L \frac{\partial i}{\partial t} dx,$$

and, on canceling dx ,

$$-\frac{\partial e}{\partial x} = iR + L \frac{\partial i}{\partial t}. \quad (36a)$$

The current $i(x,t)$ also changes from the point x to the point $x + dx$ since some of the current is diverted through the shunt conductance and capacitance. By Kirchhoff's second law, the sum of the currents flowing toward the node at x equals the sum of the currents flowing away from the node. The current flowing down through the capacitance is

$$(C dx) \frac{\partial e}{\partial t},$$

and the current flowing through the conductance is $(G dx)e$. Hence

$$i(x,t) = (C dx) \frac{\partial e}{\partial t} + (G dx) e + i(x,t) + \frac{\partial i}{\partial x} dx,$$

or

$$-\frac{\partial i}{\partial x} dx = Ge dx + C \frac{\partial e}{\partial t} dx,$$

and on dividing through by dx ,

$$-\frac{\partial i}{\partial x} = eG + C \frac{\partial e}{\partial t}. \quad (36b)$$

Equations (36a) and (36b) form a *pair* of partial differential equations. Both equations involve both unknown functions $e(x,t)$ and $i(x,t)$. It is a simple matter in this case to eliminate i (or e) and obtain an equation on e (or i) alone. This is effected by differentiating Equation (36a) with respect to x ,

$$\begin{aligned} -\frac{\partial^2 e}{\partial x^2} &= R \frac{\partial i}{\partial x} + L \frac{\partial^2 i}{\partial x \partial t} \\ &= -RG e - RC \frac{\partial e}{\partial t} + L \frac{\partial^2 i}{\partial x \partial t} \end{aligned} \quad (37)$$

(on substituting Equation (36b) for $\partial i/\partial x$) and differentiating Equation (36b) with respect to t ,

$$-\frac{\partial^2 i}{\partial x \partial t} = G \frac{\partial e}{\partial t} + C \frac{\partial^2 e}{\partial t^2}.$$

Substituting this result in Equation (37),

$$LC \frac{\partial^3 e}{\partial t^3} + (LG + RC) \frac{\partial e}{\partial t} + RGe = \frac{\partial^2 e}{\partial x^2}. \quad (38a)$$

In a similar fashion we can eliminate e and obtain

$$CL \frac{\partial^3 i}{\partial t^3} + (RC + LG) \frac{\partial i}{\partial t} + GRi = \frac{\partial^2 i}{\partial x^2}. \quad (38b)$$

Special forms of the transmission line equations can be obtained by making various assumptions regarding the parameters G , R , L , and C . If we assume that the resistance and conductance are negligible, which is reasonable at high frequencies, then Equations (38a) and (38b) reduce to

$$LC \frac{\partial^3 e}{\partial t^3} = \frac{\partial^2 e}{\partial x^2}, \quad CL \frac{\partial^3 i}{\partial t^3} = \frac{\partial^2 i}{\partial x^2}, \quad (39)$$

which are called the *high-frequency line equations*, and are of the same form as the one-dimensional wave equation.

If we assume that L and C are negligible, which is reasonable in the steady state at very low frequencies, we obtain the case of a *direct current transmission line*, namely,

$$\frac{\partial^2 e}{\partial x^2} = RGe, \quad \frac{\partial^2 i}{\partial x^2} = GRi. \quad (40)$$

Note that these are now *ordinary* differential equations.

Finally, for a submarine cable we may assume $L = 0 = G$ since the leakage conductance is very small and the frequencies are low enough to make the series inductance negligible. Thus we obtain the *submarine cable equations*,

$$RC \frac{\partial e}{\partial t} = \frac{\partial^2 e}{\partial x^2}, \quad RC \frac{\partial i}{\partial t} = \frac{\partial^2 i}{\partial x^2}, \quad (41)$$

which are of the same form as the heat equation in one dimension.

10. Potential

Another important equation of mathematical physics may be derived by considering the potential of the forces of attraction of two masses. If M and m are two point masses r units apart (see Figure 15), Newton's law states that these point masses attract

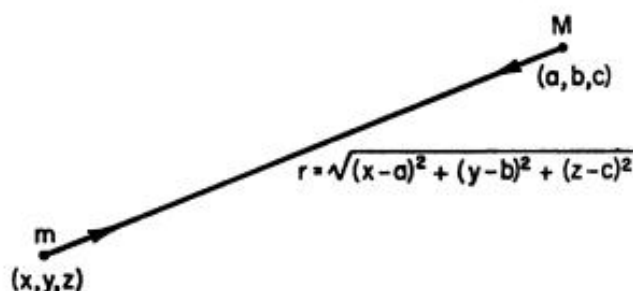


Fig. 15

each other with a force f proportional to the product of their masses and inversely proportional to the square of the distance between them,

$$f = G \frac{mM}{r^2},$$

where G is the constant of proportionality. In particular, if m is a unit mass,

$$f = \frac{GM}{r^2}.$$

Consider the function $u = GM/r$, called the *force function*, or its negative $-GM/r$, called the *potential function*. The derivative of the potential function with respect to r is precisely the force f . We shall show that the potential function satisfies a certain partial differential equation.

Instead of considering a single mass M we shall consider a number of masses $M_1, M_2, \dots, M_n$ and hence a force function

$$u = G \sum_{k=1}^n \frac{M_k}{r_k}, \quad (42)$$

where, if the coordinates of the k th mass are (a_k, b_k, c_k) and the coordinates of the unit mass are (x, y, z) ,

$$r_k = \sqrt{(x - a_k)^2 + (y - b_k)^2 + (z - c_k)^2}.$$

Equation (42) can be generalized. In the case of a *continuous* distribution of masses in some volume V ,

$$u(x,y,z) = G \iiint_V \frac{\rho(a,b,c)}{r} dV \quad (43)$$

where $\rho(a,b,c)$ is the density of the mass distribution, dV is an element of volume, and

$$r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}.$$

The partial differential equation satisfied by $u(x,y,z)$ is derived from Equation (43) as follows. Differentiating u with respect to x , we have

$$\begin{aligned} \frac{\partial u}{\partial x} &= -G \iiint_V \rho(a,b,c) \frac{(x-a)}{r^3} dV, \\ \frac{\partial^2 u}{\partial x^2} &= -G \iiint_V \rho(a,b,c) \left[\frac{1}{r^3} - 3 \frac{(x-a)^2}{r^5} \right] dV. \end{aligned}$$

If we similarly calculate $\partial^2 u / \partial y^2$ and $\partial^2 u / \partial z^2$ and add,

$$\nabla^2 u = -G \iiint_V \rho(a,b,c) \left(\frac{3}{r^3} - \frac{3}{r^3} \right) dV = 0.$$

Hence the force function $u(x,y,z)$ (or its negative, the potential function) satisfies the three-dimensional Laplacian,

$$\nabla^2 u = 0. \quad (44)$$

In the theory of electrostatics, Coulomb's law states that the force attracting or repelling two small charged bodies is

$$F = K \frac{qQ}{r^2},$$

where q and Q are the charges in coulombs on each body, respectively, r is the distance between them, and K is a constant of proportionality whose value depends on the medium in which the force is exerted. Since this law is of the same form as Newton's law for masses, the *electrostatic potential* ϕ must also satisfy the three-dimensional Laplacian,

$$\nabla^2 \phi = 0.$$

The same law applies to magnetostatics. Two magnetic poles of strength m and M , respectively, attract (or repel) each other with a force directly proportional to the magnitude of their magnetic charges and inversely proportional to the square of the distance r between them,

$$F = k \frac{mM}{r^2}.$$

The constant of proportionality k depends on the medium in which the magnetic charges are embedded. Hence the *magnetic potential* ψ also satisfies

$$\nabla^2 \psi = 0.$$

11. The equation of continuity

The so-called *equation of continuity*, which essentially states the conservation of mass, plays an important role in the theory of hydrodynamics. If a fluid (liquid or gas) is enclosed in some container V , a vectorial velocity can be associated with each point in the fluid with components u , v , w in the x -, y -, and z -directions, respectively. The equation of continuity is satisfied by these components.

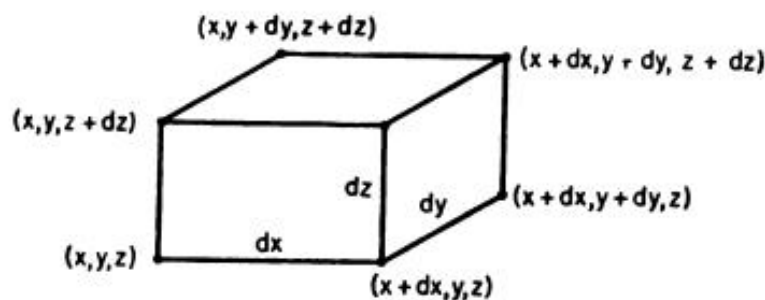


Fig. 16

Consider a small volume $dx dy dz$ of the fluid (Figure 16) and let $\rho = \rho(x, y, z, t)$ be the density of the fluid at any point (x, y, z) at any time t . The mass crossing the surface $dy dz$ (perpendicular to dx at the point x) per unit time is

$$u(\rho dy dz),$$

while the mass crossing the surface $dy dz$ at the point $x + dx$ in the x -direction is

$$\rho u dy dz + \frac{\partial}{\partial x}(\rho u) dx dy dz.$$

Hence the net gain in mass per unit time from the flow in the x -direction is

$$-\frac{\partial}{\partial x}(\rho u) dx dy dz.$$

Similarly the net gain in mass due to flow in the y - and z -directions is

$$-\frac{\partial}{\partial y}(\rho v) dx dy dz \quad \text{and} \quad -\frac{\partial}{\partial z}(\rho w) dx dy dz,$$

respectively. The total accumulation of mass per unit of time is the sum of these three contributions, namely,

$$-\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) \right] dx dy dz. \quad (45)$$

But the total mass in the elementary volume is

$$\rho dx dy dz,$$

and the rate change of this mass is

$$\frac{\partial \rho}{\partial t} dx dy dz.$$

Equating this rate change of mass to that given by Equation (45) and canceling $dx dy dz$, we obtain the *equation of continuity*,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0. \quad (46)$$

If ρ is a function of t alone and not of x, y, z , we obtain the simpler equation

$$\frac{\partial \rho}{\partial t} + \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0.$$

On the other hand, if ρ is a function of x, y, z and not a function of t , Equation (46) becomes

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0. \quad (47)$$

Finally, if ρ is a constant, the equation of continuity takes the special form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (48)$$

Equations (46), (47), and (48) involve three unknown variables, viz., u , v , and w . These equations are used in conjunction with the *equations of motion* of a fluid (Section 14), which give us three additional equations for u , v , w , and the pressure p . Hence we see that the equations of hydrodynamics form a system of differential equations. Theoretically v , w , and p could be eliminated to obtain an equation on u alone. This situation has been met before in the case of the transmission line equations.

12. Tidal waves in an incompressible fluid

The motion of an incompressible fluid in a long channel furnishes another example of an equation in fluid mechanics. The derivation of this equation involves some new techniques not encountered in previous examples—yet leads to the familiar wave equation in one dimension.

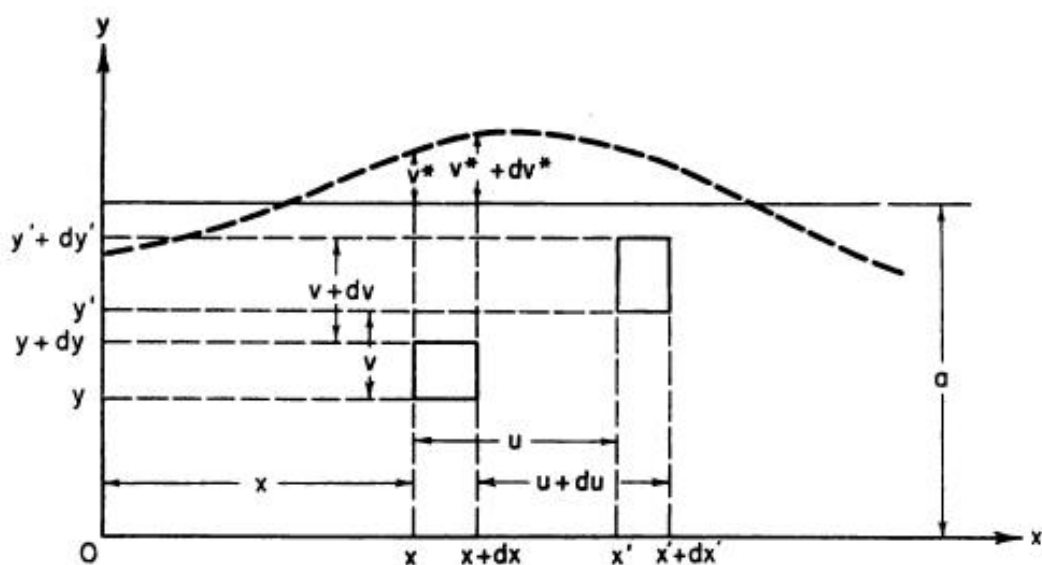


Fig. 17

A long channel of width h is oriented along the x -axis, Figure 17, and is filled to a height a by an incompressible fluid (for example, water). The fluid is disturbed by waves (as indicated by the dotted line in Figure 17) whose wavelength is assumed very long in comparison with the depth a of the fluid in the channel. Let $u(x, y)$ and $v(x, y)$ denote the components of displacement of a

point (x, y) in the x - and y -directions, respectively. We wish to derive the partial differential equation satisfied by the displacement u . Due to the wave disturbance an element of volume

$$h \, dx \, dy$$

at (x, y) will be transformed into the element

$$h \, dx' \, dy'$$

at (x', y') (see Figure 17). Now

$$x' = x + u$$

and

$$x' + dx' = x + dx + u + du = x + dx + u + \frac{\partial u}{\partial x} dx.$$

Hence

$$dx' = dx + \frac{\partial u}{\partial x} dx.$$

Similarly,

$$dy' = dy + \frac{\partial v}{\partial y} dy$$

and

$$dx' \, dy' = dx \, dy \left(1 + \frac{\partial u}{\partial x} \right) \left(1 + \frac{\partial v}{\partial y} \right).$$

On the assumption of "long" waves the slopes $\partial u / \partial x$ and $\partial v / \partial y$ will be small, and hence to first-order approximation (neglecting second-order terms),

$$dx' \, dy' = dx \, dy \left(1 + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right). \quad (49)$$

But in an incompressible fluid the volume of an element cannot change, so that

$$h \, dx' \, dy' = h \, dx \, dy.$$

We conclude, therefore, from Equation (49) that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \quad (50)$$

From the assumption that the waves are very long compared with the depth of the fluid in the channel we are justified in assuming that the horizontal displacement is independent of the depth y .

Hence, by integrating Equation (50),

$$v = -y \frac{\partial u}{\partial x} + A, \quad (51)$$

where A is a constant of integration. But at $y = 0$, $v = 0$ since there can be no vertical displacement at the bottom of the channel. Hence $A = 0$. Let v^* be the component of vertical displacement at the surface (see Figure 17). Then, from Equation (51),

$$v^* = -a \frac{\partial u}{\partial x}. \quad (52)$$

The pressure on the surface $h \, dy$ at the point $x + dx$ is greater than that on the surface $h \, dy$ at the point x since the former surface is at a greater depth. This pressure excess dp is

$$dp = \delta g \, dv^*,$$

where δ is the density of the fluid and g is the gravitational constant. But

$$dv^* = \frac{\partial v^*}{\partial x} dx.$$

Hence

$$dp = \delta g \frac{\partial v^*}{\partial x} dx.$$

The force in the x -direction on the surface $h \, dy$ at the point x exceeds that on the surface $h \, dy$ at $x + dx$ by the amount

$$- \delta g \frac{\partial v^*}{\partial x} dx (h \, dy).$$

From the fundamental law of mechanics, $\Sigma F_x = ma_x$,

$$(\delta h \, dx \, dy) \frac{\partial^2 u}{\partial t^2} = - \delta g \frac{\partial v^*}{\partial x} h \, dx \, dy$$

or

$$\frac{\partial^2 u}{\partial t^2} = -g \frac{\partial v^*}{\partial x}. \quad (53)$$

Substituting $\partial v^*/\partial x$ from Equation (52) in the above equation we obtain the desired result, viz.,

$$\frac{\partial^2 u}{\partial t^2} = g a \frac{\partial^2 u}{\partial x^2}, \quad (54)$$

which we immediately recognize as the wave equation in one dimension.

13. The equations of elasticity

Numerous partial differential equations appear in the theory of elasticity, but their derivation requires a specialized knowledge of the subject. We shall limit our treatment here to the derivation of one type of equation which can be deduced almost entirely from the principles of elementary mechanics. Other equations of elasticity will be found in Section 14.

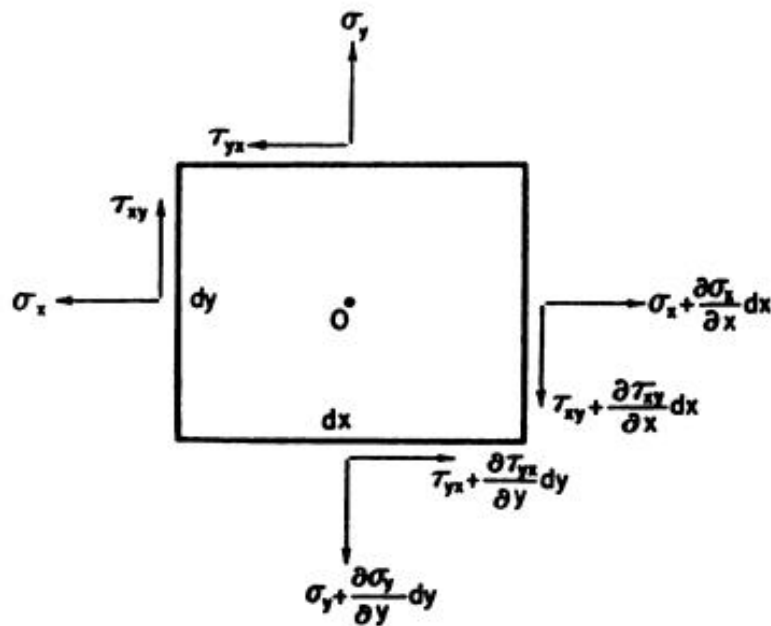


Fig. 18

Consider a thin elastic plate upon which no body forces are acting. On each side of an elementary rectangle, $dx\,dy$ (Figure 18) normal components of stress σ and tangential components of shear τ will act. Since the element is assumed to be in equilibrium, the sum of the forces in the x - and y -directions, ΣF_x and ΣF_y , must be zero. Hence

$$\begin{aligned}\Sigma F_x = & -\sigma_x dy + \left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx \right) dy - \tau_{yx} dx \\ & + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy \right) dx = 0\end{aligned}$$

or

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = 0. \quad (55)$$

Similarly, the equation $\Sigma F_y = 0$ leads to

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0. \quad (56)$$

Equations (55) and (56) are known as the *equations of equilibrium*. Equating to zero the sum of the moments ΣM_0 about the center of the rectangle $dx \, dy$, we have

$$\begin{aligned} (\tau_{yz} \, dx) \frac{dy}{2} + \left[\left(\tau_{yz} + \frac{\partial \tau_{yz}}{\partial y} dy \right) dx \right] \frac{dy}{2} \\ - (\tau_{xy} \, dy) \frac{dx}{2} - \left[\left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x} dx \right) dy \right] \frac{dx}{2} = 0, \end{aligned} \quad (57)$$

or, neglecting infinitesimals of higher order and dividing by $dx \, dy$,

$$\tau_{xy} = \tau_{yx}.$$

The equations of equilibrium then become

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0, \quad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0. \quad (58)$$

The two Equations (58) involve three unknowns σ_x , σ_y , τ_{xy} . So far we have utilized only the mechanics of rigid bodies. To find a third equation we must also consider the elastic deformation of the body. A statement of the fact that the elastic body remains continuous after deformation leads to the equation

$$\frac{\partial^2}{\partial x^2} (\sigma_x + \sigma_y) + \frac{\partial^2}{\partial y^2} (\sigma_x + \sigma_y) = 0, \quad (59)$$

known as the *equation of compatibility* (see [28]).

One of the techniques used in solving these equations (two of equilibrium and the equation of compatibility) consists in introducing a function $\phi(x, y)$ called the *Airy stress function*, with the properties that

$$\sigma_x = \frac{\partial^2 \phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}. \quad (60)$$

If σ_x , σ_y , and τ_{xy} are given in terms of such a function, the equations of equilibrium are automatically satisfied, for

$$\frac{\partial}{\partial x} \left(\frac{\partial^2 \phi}{\partial y^2} \right) - \frac{\partial}{\partial y} \left(\frac{\partial^2 \phi}{\partial x \partial y} \right) = 0, \quad \frac{\partial}{\partial y} \left(\frac{\partial^2 \phi}{\partial x^2} \right) - \frac{\partial}{\partial x} \left(\frac{\partial^2 \phi}{\partial x \partial y} \right) = 0.$$

In terms of ϕ the equation of compatibility becomes

$$\frac{\partial^2}{\partial x^2} \left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2} \right) + \frac{\partial^2}{\partial y^2} \left(\frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial x^2} \right) = 0$$

or

$$\frac{\partial^4 \phi}{\partial x^4} + 2 \frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0.$$

The operator

$$\nabla^4 = \nabla^2(\nabla^2) = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4},$$

that is, the operator ∇^2 operating on ∇^2 is the square of the Laplacian in two dimensions. The equation satisfied by the Airy stress function may therefore be written as

$$\nabla^4 \phi = 0, \quad (61)$$

and is known as the *biharmonic equation*. Once the equation $\nabla^4 \phi = 0$ is solved subject to the given boundary conditions, σ_x , σ_y , τ_{xy} can be determined from Equations (60).

14. Additional equations

Since many partial differential equations cannot be derived without a specialized knowledge of particular branches of mathematical physics, their derivation would be out of place in a book of this nature. However, for the sake of completeness some of the more important equations appearing in the various branches of engineering will be listed here. References where more detailed information concerning these various equations may be found are given in connection with each equation.

We have considered only *linear* partial differential equations, that is, equations in which the unknown function and its derivatives appear only to the first power. However, the reader should not infer from this that nonlinear equations are unimportant; many of the equations of hydrodynamics and relativity fall into this category.

1. *Equations of Compatibility in Three Dimensions :*

$$(1 + \nu) \nabla^2 \sigma_x + \frac{\partial^2}{\partial x^2} (\sigma_x + \sigma_y + \sigma_z) = 0,$$

$$(1 + \nu) \nabla^2 \sigma_y + \frac{\partial^2}{\partial y^2} (\sigma_x + \sigma_y + \sigma_z) = 0,$$

$$(1 + \nu) \nabla^2 \sigma_z + \frac{\partial^2}{\partial z^2} (\sigma_x + \sigma_y + \sigma_z) = 0,$$

$$(1 + \nu) \nabla^2 \tau_{yz} + \frac{\partial^2}{\partial y \partial z} (\sigma_x + \sigma_y + \sigma_z) = 0,$$

$$(1 + \nu) \nabla^2 \tau_{xz} + \frac{\partial^2}{\partial x \partial z} (\sigma_x + \sigma_y + \sigma_z) = 0,$$

$$(1 + \nu) \nabla^2 \tau_{xy} + \frac{\partial^2}{\partial x \partial y} (\sigma_x + \sigma_y + \sigma_z) = 0,$$

where ν is Poisson's ratio, σ_x , σ_y , σ_z are the normal stress components, and τ_{yz} , τ_{xz} , τ_{xy} are the tangential stress components (see [27], [28]).

2. *Bending of an Elastic Plate :*

$$\frac{\partial^4 u}{\partial x^4} + 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = \frac{q}{D}, \quad (62)$$

where u is the deflection, q is the loading per unit area, D is the flexural rigidity (see [29]).

3. *Buckling of an Elastic Plate :*

$$\begin{aligned} & \frac{\partial^4 u}{\partial x^4} + 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} \\ &= \frac{1}{D} \left(N_x \frac{\partial^2 u}{\partial x^2} + N_y \frac{\partial^2 u}{\partial y^2} + 2 N_{xy} \frac{\partial^2 u}{\partial x \partial y} \right), \end{aligned} \quad (63)$$

where u is the deflection, D is the flexural rigidity, N_x , N_y are the normal forces components, and N_{xy} is the tangential shear component per unit of length (see [29]).

4. *Torsion of a Prismatical Bar :*

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -2, \quad (64)$$

where ϕ is the stress function $\partial\phi/\partial y = \tau_{xz}$, $\partial\phi/\partial x = -\tau_{yz}$, where τ_{xz} and τ_{yz} are the x - and y -components of the tangential stress (see [27]).

5. *Equations of Motion of a Fluid :*

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} &= X - \frac{1}{\delta} \frac{\partial p}{\partial x}, \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} &= Y - \frac{1}{\delta} \frac{\partial p}{\partial y}, \\ \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} &= Z - \frac{1}{\delta} \frac{\partial p}{\partial z}, \end{aligned}$$

where X, Y, Z are the components of the external force, u, v, w are the components of velocity, p is the pressure, and δ is the density (see [16]).

6. *Laminar Flow Equations :*

$$\begin{aligned} u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} &= -\frac{1}{\delta} \frac{\partial p}{\partial x} + \frac{\mu}{\delta} \nabla^2 u, \\ u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} &= -\frac{1}{\delta} \frac{\partial p}{\partial y} + \frac{\mu}{\delta} \nabla^2 v, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \end{aligned}$$

where u, v are the components of velocity, δ is the density, p is the pressure, and μ is the coefficient of viscosity (see [16]).

7. *Dynamical Equations for an Incompressible Fluid :*

$$\begin{aligned} \delta \frac{\partial u}{\partial t} &= \delta X - \frac{\partial p}{\partial x} + \mu \nabla^2 u, \\ \delta \frac{\partial v}{\partial t} &= \delta Y - \frac{\partial p}{\partial y} + \mu \nabla^2 v, \\ \delta \frac{\partial w}{\partial t} &= \delta Z - \frac{\partial p}{\partial z} + \mu \nabla^2 w, \end{aligned}$$

where X, Y, Z are the components of the external force, u, v, w are the components of velocity, p is the pressure, δ is the density, and μ is the coefficient of viscosity (see [16]).

8. *Vibration of a Microphone Diaphragm :*

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{x} \frac{\partial u}{\partial x} - \left(a \frac{\partial^2 u}{\partial t^2} + b \frac{\partial u}{\partial t} \right) = 0, \quad (65)$$

where u is the displacement of a diaphragm in a capacitor microphone, and a and b are constants (see [18], [19]).

9. *Oscillation in a Loud Speaker Horn :*

$$\frac{\partial^2 u}{\partial x^2} + \beta \frac{\partial u}{\partial x} - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0, \quad (66)$$

where u is the velocity potential, c is the speed of sound, and β is a constant (see [18], [19]).

10. *Maxwell's Equations :*

$$\begin{aligned} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= -\frac{1}{c} \frac{\partial B_x}{\partial t}, \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} &= -\frac{1}{c} \frac{\partial B_y}{\partial t}, \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= -\frac{1}{c} \frac{\partial B_z}{\partial t}, \\ \frac{\partial H_z}{\partial t} - \frac{\partial H_y}{\partial z} &= \frac{4\pi\gamma}{c} E_x + \frac{1}{c} D_x, \\ \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} &= \frac{4\pi\gamma}{c} E_y + \frac{1}{c} D_y, \\ \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= \frac{4\pi\gamma}{c} E_z + \frac{1}{c} D_z, \\ \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} &= 0, \\ \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} &= 4\pi\rho, \end{aligned}$$

where E_x, E_y, E_z are the components of electric field strength, H_x, H_y, H_z are the components of magnetic intensity, D_x, D_y, D_z

are the components of electric flux density, B_x , B_y , B_z are the components of magnetic flux density, c is the speed of light, γ is the conductivity, and ρ is the charge density (see [26]).

PROBLEMS

1. Verify the dimensions of a in the equation

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

for the vibrating string and prove that the equation is dimensionally correct.

2. Where, in the derivation of the equation of the vibrating string (Section 2), were the various "idealized" assumptions used?

3. Write the boundary conditions for the vibrating string under the conditions that the string is initially undisplaced, fixed at both end points, tightly stretched, and given an initial velocity

$$V_0 \sin \frac{\pi x}{L}.$$

(L is the length of the string.)

4. A string of length L has its right end fixed, while the left end point is given a displacement $f(t)$, starting from rest. Write the corresponding boundary conditions.

5. Assume that the string is fixed at both end points and given an initial displacement as in Figure 19 below. Determine the appropriate boundary conditions.

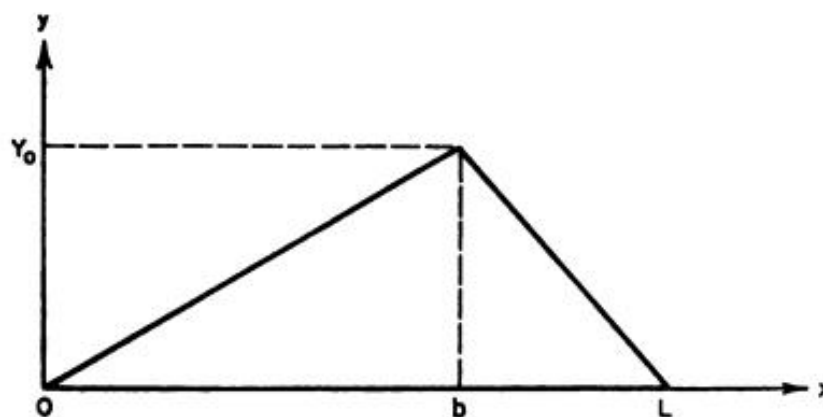


Fig. 19

6. Check the dimensions of the equation of the vibrating membrane and of the vibrating elastic solid.

7. A membrane has the shape of a square of side a units. It is given an initial displacement $u = xy(x - a)(y - a)$ and has no initial velocity. Write the boundary conditions for the transverse vibrations of the membrane.

8. Determine the boundary conditions for the longitudinal vibrations of an elastic beam if the initial displacement of the beam is $f(x)$, the initial velocity is $g(x)$, and the beam is built-in.

9. Derive the equation for the longitudinal vibrations of a bar under the assumption that the cross-sectional area varies linearly with the length of the bar.

10. If the initial displacement of a cantilever beam is a given function of x , say $\phi(x)$, and the initial velocity $\psi(x)$, determine the boundary conditions satisfied by the longitudinal vibrations of the beam.

11. Consider a shaft of length L which vibrates torsionally. Under the assumption that all cross sections of the shaft remain plane during rotation, derive the equation satisfied by the angle of twist θ . Assume the cross section of the shaft to be a circle of area A .

12. Determine the boundary conditions satisfied by the angle of twist of a shaft vibrating in torsion under the assumption that at $t = 0$ the angle of twist is a known function of x , the angular velocity is a known function of x , and the shaft is built-in at both ends.

13. Write the boundary conditions for the torsional vibrations of a shaft as in Problem 12, but assume that the shaft is built-in at the end $x = 0$ and is free at the end $x = L$.

14. Check the dimensions of the equation for the transverse vibrations of a beam (Section 6).

15. Let a beam of length L , built-in at $x = 0$ and free at the end $x = L$, be subjected to a force $F = F_0 \sin t$ at the end $x = L$. Determine the boundary conditions satisfied by the transverse displacement of the beam.

16. If the beam of Problem 15 is free of external forces, but its end $x = 0$ is subjected to a motion $y = Y_0 \sin t$, write the boundary conditions satisfied by the transverse motion of the beam.

17. The force per unit area on the end $x = 0$ of a semi-infinite bar is $f(t)$. If the bar is fixed at $x = \infty$ and the initial displacement and velocity of each section are zero, set up the differential equation and boundary conditions for the longitudinal displacement $u(x, t)$ of the bar.

18. Consider a uniform circular disk whose entire surface is insulated. Assume that its temperature at $t = 0$ is a function only of the distance r from the center of the disk. Starting from basic physical laws, show that the temperature $u(r, t)$ of the disk satisfies the equation

$$\frac{\partial u}{\partial t} = K \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right).$$

19. A thin bar is conducting heat along its length and is also radiating heat from its surface. The temperature of the surroundings is T . Assume that the radiation obeys Newton's law, that is, the time rate of change of temperature due to radiation is proportional to the temperature difference between the body and its surroundings. The bar has a cross-sectional area A and a density δ . Derive the differential equation satisfied by the temperature.

20. Consider the flow of heat in a long thin bar (one-dimensional case). If the bar gains heat due to radioactive decay, determine the equation satisfied by the temperature $u(x, t)$. Assume that the rate at which heat is gained per unit volume by radioactive decay is proportional to e^{-ax} .

21. Consider the one-dimensional heat equation in the case where heat is gained by the decomposition of the material in the bar. Derive the differential equation satisfied by the temperature if the rate at which heat is gained per unit volume by decomposition is a constant, A .

22. Derive the equation of the vibrating string in the case in which an external normal force of $F(x)$ lbs/ft acts on the string.

23. Write the boundary conditions for the one-dimensional heat equation if the ends of the bar (of length L), instead of being held at a constant temperature, have no temperature gradient across their surfaces (adiabatic case). Assume that the initial temperature distribution is $f(x)$.

24. Derive the telegraph equations for the case of a direct current transmission line, that is, one in which the inductance and capacitance may be neglected.

25. An electric transmission line of length l is initially in the relaxed state, that is, the currents and voltages in the line are zero. At $t = 0$,

a voltage $E = E_0 \sin \omega t$ is applied to the sending end ($x = 0$). The other end of the line is connected to a pure resistance R . Write the boundary conditions satisfied by the voltage $e(x, t)$.

26. It was shown in Section 10 that the function $1/r$, where

$$r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2},$$

satisfied the three-dimensional Laplacian,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Show that the two-dimensional Laplacian

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is satisfied by the function $\log 1/r$, where

$$r = \sqrt{(x-a)^2 + (y-b)^2}.$$

27. Derive the equation of continuity of hydrodynamics on the assumption that the density ρ varies linearly with x and is independent of time and the y - and z -coordinates.

28. A horizontal cantilever beam of length L and slender rectangular cross section carries a uniformly distributed load on its upper surface (see Figure 20). Determine the boundary conditions satisfied by the stresses σ_x , σ_y , and τ_{xy} .

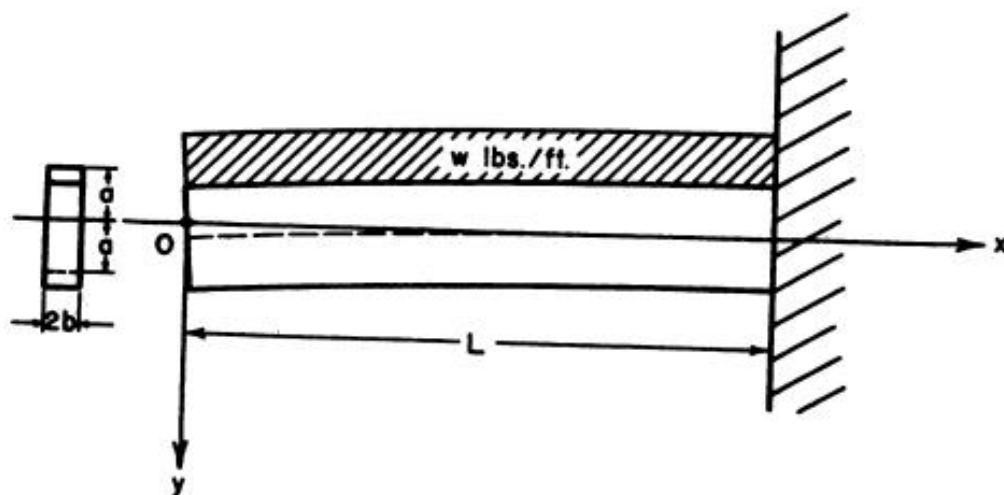


Fig. 20

Chapter II

FOURIER SERIES

15. Preliminary remarks

In the calculus the reader considered the problem of expanding a function $f(x)$ in a power series ([24], Section 5.9) and obtained the result that

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots \quad (67)$$

We chose an expansion in powers of x because these are simple functions to handle and are amenable to numerical calculations and the operations of the calculus. For example, one way to integrate e^{x^2} is to expand e^{x^2} into a power series and integrate term by term.

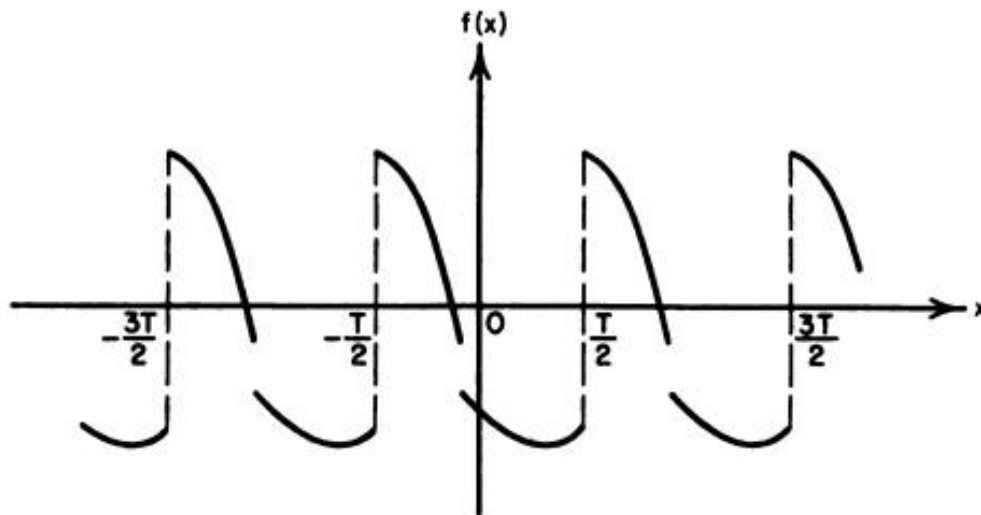


Fig. 21

In many engineering problems periodic functions are used. If $f(x)$ is a periodic function of period T , that is, $f(x + T) = f(x)$ (see Figure 21), it would appear logical to try to expand $f(x)$ in the simplest *periodic* functions, namely, sines and cosines, rather than

in powers of x . We shall therefore attempt to obtain a representation for $f(x)$ in the form

$$\begin{aligned} f(x) &= A_0 + a_1 \cos x + a_2 \cos 2x + \dots \\ &\quad + b_1 \sin x + b_2 \sin 2x + \dots \\ &= A_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx. \end{aligned}$$

Such a representation is known as a *Fourier series*, or the Fourier series expansion of the function $f(x)$.

The reasons for wishing to obtain such an expansion of a periodic function are analogous but more far-reaching than those given above for an ordinary power series. Fourier series expansions will enable us to solve various partial differential equations and will lead to important generalizations like the Fourier transform.

Our immediate problem is then to obtain an expression for the coefficients A_0 , a_n , and b_n of the Fourier series as was done in the case of power series. This requires the knowledge of certain integral formulas derived in the following section.

16. Some integral formulas

In order to proceed in the most efficient fashion it will be convenient to recall certain definite integrals,

$$\left. \begin{aligned} \int_{-\pi}^{\pi} \cos mx \cos nx \, dx &= \pi \delta_{mn} \\ \int_{-\pi}^{\pi} \sin mx \sin nx \, dx &= \pi \delta_{mn} \\ \int_{-\pi}^{\pi} \sin mx \cos nx \, dx &= 0 \end{aligned} \right\} \quad m, n \text{ integers,} \quad (68)$$

where δ_{mn} is the Kronecker delta defined as follows :

$$\begin{aligned} \delta_{mn} &= 0 & \text{if } m \neq n, \\ \delta_{mn} &= 1 & \text{if } m = n. \end{aligned}$$

The formulas above, Equations (68), are valid for all nonnegative integral values of m and n not both zero. They are known as *orthogonality conditions*. For $n = m = 0$, it is clear that

$$\int_{-\pi}^{\pi} \cos 0x \cos 0x \, dx = \int_{-\pi}^{\pi} dx = 2\pi,$$

$$\int_{-\pi}^{\pi} \sin 0x \sin 0x \, dx = \int_{-\pi}^{\pi} 0 \, dx = 0,$$

$$\int_{-\pi}^{\pi} \sin 0x \cos 0x \, dx = \int_{-\pi}^{\pi} 0 \, dx = 0.$$

Since the proof of all the above formulas rests on the trigonometric identities

$$\cos mx \cos nx = \frac{1}{2} \cos (m+n)x + \frac{1}{2} \cos (m-n)x,$$

$$\sin mx \sin nx = \frac{1}{2} \cos (m-n)x - \frac{1}{2} \cos (m+n)x,$$

$$\sin mx \cos nx = \frac{1}{2} \sin (m+n)x + \frac{1}{2} \sin (m-n)x,$$

we shall establish only the first of Equations (68).

$$\begin{aligned} \int_{-\pi}^{\pi} \cos mx \cos nx \, dx &= \frac{1}{2} \int_{-\pi}^{\pi} \cos (m+n)x \, dx + \frac{1}{2} \int_{-\pi}^{\pi} \cos (m-n)x \, dx \\ &= \frac{1}{2} \left. \frac{\sin (m+n)x}{m+n} \right|_{-\pi}^{\pi} + \frac{1}{2} \left. \frac{\sin (m-n)x}{m-n} \right|_{-\pi}^{\pi} \\ &= 0 \quad \text{if} \quad m+n \neq 0 \quad \text{and} \quad m-n \neq 0. \end{aligned}$$

But $m+n$ and $m-n$ are different from zero if m and n are distinct nonnegative integers. Hence

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \quad \text{if} \quad m \neq n.$$

If, on the other hand, $m = n$, then

$$\begin{aligned} \int_{-\pi}^{\pi} \cos mx \cos mx \, dx &= \int_{-\pi}^{\pi} \cos^2 mx \, dx \\ &= \int_{-\pi}^{\pi} \left(\frac{1 + \cos 2mx}{2} \right) dx = \pi. \end{aligned}$$

These two results establish the validity of the first of Equations (68).

17. Calculation of the coefficients

We now are in a position to derive formulas for the coefficients of the Fourier series expansion of a function $f(x)$. We shall assume,

for the present, that the function $f(x)$ is periodic of period 2π (the period of $\sin x$ and $\cos x$), and later shall extend our results to functions with periods of arbitrary length.

Let us write the Fourier series expansion of $f(x)$ in the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx. \quad (69)$$

(The reason for using $\frac{1}{2}a_0$ rather than a_0 will become apparent shortly.) Multiply both sides of Equation (69) by $\cos 0x = 1$, and integrate from $-\pi$ to $+\pi$:

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) dx &= \int_{-\pi}^{\pi} \frac{1}{2} a_0 dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos 0x dx \\ &\quad + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos 0x dx. \end{aligned}$$

But by the orthogonality conditions (Equations 68) all the terms of both summations are zero. Hence

$$\int_{-\pi}^{\pi} f(x) dx = \int_{-\pi}^{\pi} \frac{1}{2} a_0 dx = a_0 \pi$$

and

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx. \quad (70)$$

To continue this process we shall multiply both sides of Equation (69) by $\cos mx$ ($m \neq 0$) and integrate from $-\pi$ to π , viz.,

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos mx dx &= \frac{1}{2} a_0 \int_{-\pi}^{\pi} \cos mx dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx dx \\ &\quad + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos mx dx. \end{aligned}$$

By the orthogonality conditions,

$$\int_{-\pi}^{\pi} \cos mx dx = 0,$$

$$\sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \cos mx dx = \pi a_m,$$

$$\sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0,$$

that is, all terms of the Fourier series vanish except the one involving $\cos mx$. Hence

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \pi a_m,$$

and on changing m to n ,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad n = 1, 2, \dots$$

Note that since we used $\frac{1}{2}a_0$ in our Fourier series expansion, a_0 (see Equation 70) is also given by the above formulas, namely,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos 0x \, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \, dx.$$

In order to calculate the b_n 's we multiply both sides of Equation (69) by $\sin mx$ and integrate from $-\pi$ to π .

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \sin mx \, dx &= \frac{1}{2} a_0 \int_{-\pi}^{\pi} \sin mx \, dx + \sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \sin mx \, dx \\ &\quad + \sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin mx \, dx. \end{aligned}$$

By our orthogonality conditions,

$$\int_{-\pi}^{\pi} \sin mx \, dx = 0,$$

$$\sum_{n=1}^{\infty} a_n \int_{-\pi}^{\pi} \cos nx \sin mx \, dx = 0,$$

$$\sum_{n=1}^{\infty} b_n \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \pi b_m.$$

Hence

$$\int_{-\pi}^{\pi} f(x) \sin mx \, dx = \pi b_m,$$

and on changing from m to n ,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

18. Recapitulation

Summing up the results of the previous section we may say :
If $f(x)$ is a periodic function of x with period 2π , then

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx & n &= 0, 1, 2, \dots, \\ b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx & n &= 1, 2, 3, \dots. \end{aligned} \quad (71)$$

The interval $(-\pi, \pi)$ is known as the *full Fourier interval*. Since the function $f(x)$ is periodic of period 2π , once it has been defined for $-\pi < x \leq \pi$, it is determined for *all* values of x .

19. Example of a Fourier series

In this section we shall consider an example of a Fourier expansion of a given periodic function. The function to be considered is known as a *square wave* and is defined analytically in the full Fourier interval by the equations

$$\begin{aligned} f(x) &= -h & -\pi < x < 0, \\ f(x) &= +h & 0 < x < \pi, \end{aligned}$$

where h is a constant. This function is graphically represented in Figure 22. By Equations (71),

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ &= \frac{1}{\pi} \int_{-\pi}^0 (-h) \cos nx \, dx + \frac{1}{\pi} \int_0^{\pi} h \cos nx \, dx \\ &= -\frac{h}{\pi} \left(\frac{\sin nx}{n} \right) \Big|_{-\pi}^0 + \frac{h}{\pi} \left(\frac{\sin nx}{n} \right) \Big|_0^{\pi} = 0 \quad (n \neq 0), \end{aligned}$$

since $\sin 0 = \sin n\pi = \sin(-n\pi) = 0$. Hence all the a_n , $n = 1, 2, \dots$, are zero. The case $n = 0$ cannot be evaluated

by the above formula since we cannot divide by zero, but a direct calculation yields

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^0 (-h) dx + \frac{1}{\pi} \int_0^{\pi} h dx = 0.$$

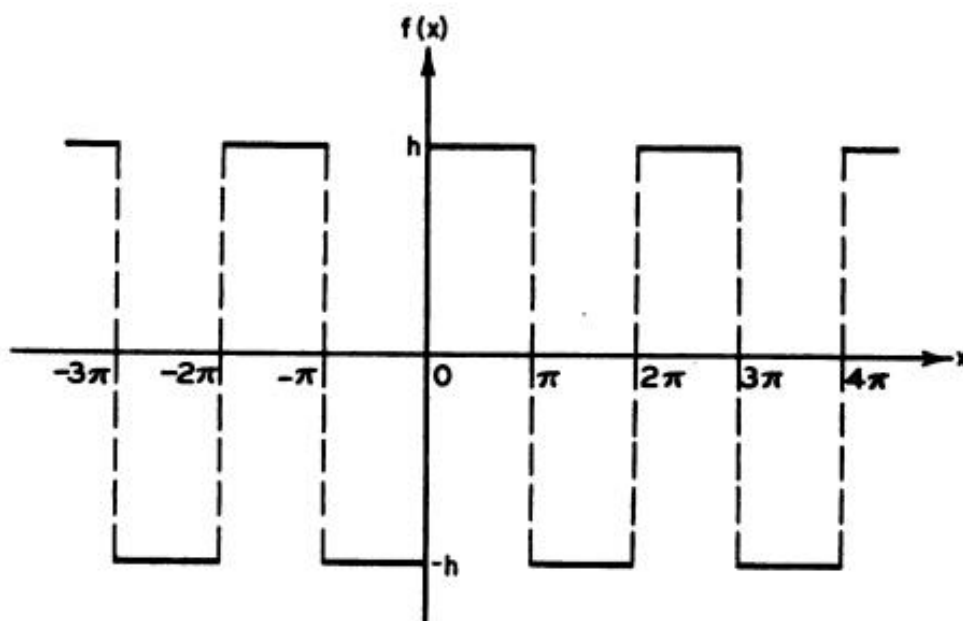


Fig. 22

To calculate the b_n 's we write

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx \\ &= \frac{1}{\pi} \int_{-\pi}^0 (-h) \sin nx dx + \frac{1}{\pi} \int_0^{\pi} h \sin nx dx \\ &= \frac{h}{n\pi} [1 - \cos(-n\pi)] - \frac{h}{n\pi} [\cos n\pi - 1] = \frac{2h}{n\pi} [1 - \cos n\pi], \end{aligned}$$

since $\cos(-\theta) = \cos \theta$. Also, since

$$\begin{aligned} \cos 0 &= +1, \\ \cos \pi &= -1, \\ \cos 2\pi &= +1, \\ \cos 3\pi &= -1, \\ \dots &\dots \dots \end{aligned}$$

and in general

$$\cos n\pi = (-1)^n,$$

we find that

$$b_n = \frac{2h}{n\pi} [1 - (-1)^n].$$

Thus if n is odd,

$$b_n = \frac{2h}{n\pi} [1 - (-1)] = \frac{4h}{n\pi},$$

if n is even,

$$b_n = \frac{2h}{n\pi} [1 - 1] = 0,$$

and we conclude therefore that

$$a_n = 0 \quad n = 0, 1, 2, \dots,$$

$$b_n = 0 \quad n = 2, 4, 6, \dots,$$

$$b_n = \frac{4h}{n\pi} \quad n = 1, 3, 5, \dots$$

Our Fourier series now reduces to

$$f(x) = \sum_{n=1,3,5,\dots}^{\infty} \frac{4h}{n\pi} \sin nx = \frac{4h}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{\sin nx}{n}. \quad (72)$$

The function $f(x)$ can also be written in the form where the dummy variable runs from one to infinity by consecutive integers by letting

$$n = 2m - 1.$$

Then if $n = 1$, $m = 1$; if $n = 3$, $m = 2$, etc. and

$$f(x) = \frac{4h}{\pi} \sum_{m=1}^{\infty} \frac{\sin (2m - 1)x}{2m - 1}. \quad (73)$$

We see from this equation that a square wave can be represented as the sum of an infinite number of sinusoidal components. It would appear logical, therefore, that if we took only a *finite* number of sine waves we should get an approximate value of $f(x)$ —the larger the number of terms the better the approximation. This question will be investigated in Section 27.

20. Change of interval

We have assumed $f(x)$ to be periodic of period 2π as indicated in Figure 23. Hence we may write the Fourier series expansion of $f(x)$ as

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

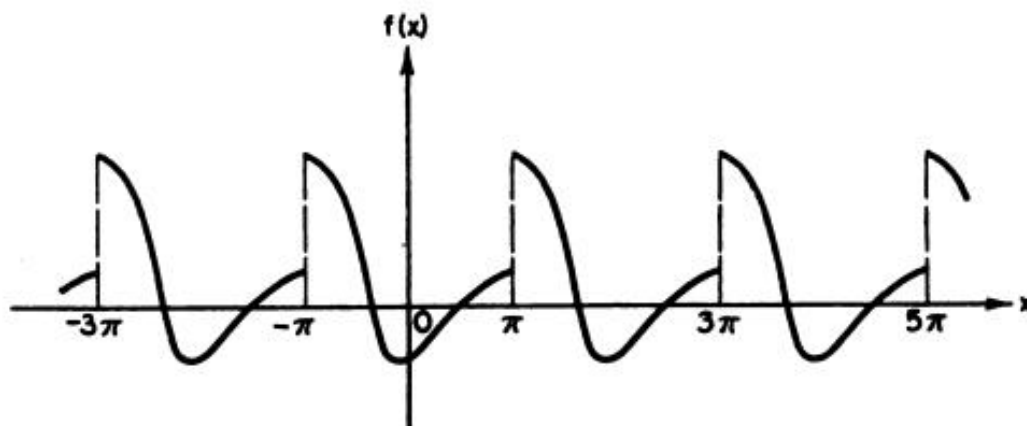


Fig. 23

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

In certain applications it is convenient to use the interval $(\alpha, \alpha + 2\pi)$ where α is any real number, rather than the interval $(-\pi, \pi)$. We shall prove in this case that the coefficients a_n and b_n are given by

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx \, dx, \\ b_n &= \frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \sin nx \, dx. \end{aligned} \tag{74}$$

Break the above integrals into three integrals, then, for example,

$$\begin{aligned} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx \, dx &= \int_{\alpha}^{-\pi} f(x) \cos nx \, dx + \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ &\quad + \int_{\pi}^{\alpha+2\pi} f(x) \cos nx \, dx. \end{aligned} \tag{75}$$

In the last integral on the right make the change of variable

$$x = y + 2\pi.$$

Then

$$\begin{aligned}\int_{\alpha}^{\alpha+2\pi} f(x) \cos nx \, dx &= \int_{-\pi}^{\alpha} f(y+2\pi) \cos n(y+2\pi) \, dy \\ &= \int_{-\pi}^{\alpha} f(y) \cos ny \, dy,\end{aligned}$$

since the function f is periodic of period 2π . On noting that the second integral of Equation (75) is precisely πa_n and that

$$\int_{\alpha}^{-\pi} = - \int_{-\pi}^{\alpha},$$

we obtain the result that

$$\frac{1}{\pi} \int_{\alpha}^{\alpha+2\pi} f(x) \cos nx \, dx = a_n.$$

A similar proof holds for b_n .

Commonly used values of α are 0 and $-\pi$. In the case $\alpha = 0$ we obtain.

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx \, dx,$$

$$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx \, dx,$$

while in the case $\alpha = -\pi$ we obtain the standard formulas of Equations (71).

21. Example of change of interval

To illustrate the gain in simplicity that can be achieved by the use of the formulas of the last section, consider the problem of expanding the function $f(x)$ given graphically in Figure 24 and analytically by the equations

$$f(x) = \frac{1}{\pi} [x + (2\pi - 1)], \quad -2\pi + 1 < x < 1. \quad (76)$$

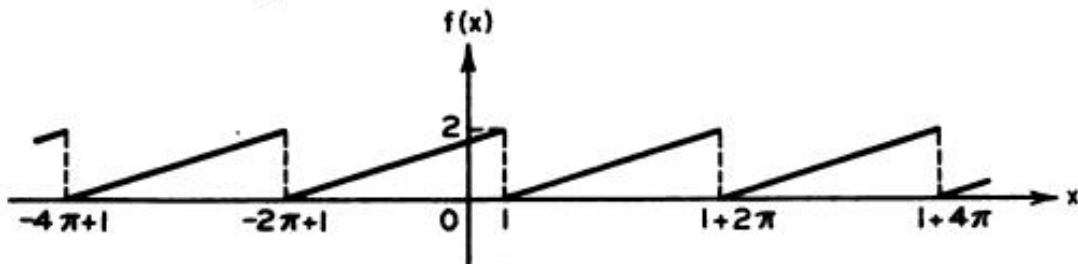


Fig. 24

Of course we could define $f(x)$ in the full Fourier interval $(-\pi, \pi)$ by the two equations

$$\begin{aligned} f(x) &= \frac{1}{\pi} [x + (2\pi - 1)] & -\pi < x < 1, \\ f(x) &= \frac{1}{\pi} (x - 1) & 1 < x < \pi, \end{aligned}$$

but Equation (76) is a simpler representation since it involves only one analytical expression. If we let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

we may write by Equations (74) with $\alpha = -2\pi + 1$,

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-2\pi+1}^1 f(x) \cos nx \, dx \\ &= \frac{1}{\pi} \int_{-2\pi+1}^1 \frac{1}{\pi} (x + 2\pi - 1) \cos nx \, dx, \\ b_n &= \frac{1}{\pi} \int_{-2\pi+1}^1 f(x) \sin nx \, dx \\ &= \frac{1}{\pi} \int_{-2\pi+1}^1 \frac{1}{\pi} (x + 2\pi - 1) \sin nx \, dx. \end{aligned}$$

With the aid of a table of integrals (see [10]) we readily compute a_n and b_n :

$$\begin{aligned} a_n &= \frac{1}{\pi^2} \left(\frac{\cos nx}{n^2} + \frac{x \sin nx}{n} \right) \Big|_{-2\pi+1}^1 + \frac{2\pi-1}{\pi^2} \left(\frac{\sin nx}{n} \right) \Big|_{-2\pi+1}^1 \\ &= \frac{2 \sin n}{\pi n}, \quad n \neq 0, \end{aligned}$$

$$a_0 = 2,$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi^2} \left(\frac{\sin nx}{n^2} - \frac{x \cos nx}{n} \right) \Big|_{-2\pi+1}^1 + \frac{2\pi-1}{\pi^2} \left(-\frac{\cos nx}{n} \right) \Big|_{-2\pi+1}^1 \\ &= -\frac{2 \cos n}{\pi n}, \quad n = 1, 2, 3, \dots \end{aligned}$$

Hence the Fourier series expansion of the function $f(x)$ of Figure 24 is given by

$$f(x) = 1 + \frac{2}{\pi} \sum_{n=1}^{\infty} \left(\frac{\sin \pi}{n} \cos nx - \frac{\cos \pi}{n} \sin nx \right). \quad (77)$$

22. Even and odd functions

A function $f(x)$ is said to be *even* if

$$f(-x) = f(x), \quad (78)$$

and is said to be *odd* if

$$f(-x) = -f(x). \quad (79)$$

For example, $\cos x$ is an even function, since

$$\cos(-x) = \cos x,$$

and $\sin x$ is an odd function, since

$$\sin(-x) = -\sin x.$$

In the cases in which the function we wish to expand into a Fourier series is even or odd, we can deduce simpler forms for its Fourier coefficients a_n and b_n . Suppose $f(x)$ to be even, then

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\ &= \frac{1}{\pi} \int_{-\pi}^0 f(x) \cos nx \, dx + \frac{1}{\pi} \int_0^{\pi} f(x) \cos nx \, dx. \end{aligned}$$

In the first integral on the right let $x = -z$. Then

$$\begin{aligned} \frac{1}{\pi} \int_{-\pi}^0 f(x) \cos nx \, dx &= \frac{1}{\pi} \int_{\pi}^0 f(-z) \cos n(-z) d(-z) \\ &= \frac{1}{\pi} \int_0^{\pi} f(z) \cos nz \, dz, \end{aligned}$$

since interchanging the limits of integration changes the sign of the integral, $f(-z) = f(z)$ and $\cos nz = \cos(-nz)$. If we change the dummy variable from z to x , we may write a_n as

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_0^{\pi} f(x) \cos nx \, dx + \frac{1}{\pi} \int_0^{\pi} f(x) \cos nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx. \end{aligned}$$

Similarly, the integral for b_n becomes

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\ &= \frac{1}{\pi} \int_{-\pi}^0 f(x) \sin nx \, dx + \frac{1}{\pi} \int_0^{\pi} f(x) \sin nx \, dx. \end{aligned}$$

As before, let $x = -z$ in the first integral on the right. Then

$$\begin{aligned} \frac{1}{\pi} \int_{-\pi}^0 f(x) \sin nx \, dx &= \frac{1}{\pi} \int_{\pi}^0 f(-z) \sin n(-z) \, d(-z) \\ &= -\frac{1}{\pi} \int_0^{\pi} f(z) \sin nz \, dz \end{aligned}$$

since $f(-z) = f(z)$ and $\sin nz = -\sin n(-z)$. We conclude therefore that

$$b_n = -\frac{1}{\pi} \int_0^{\pi} f(z) \sin nz \, dz + \frac{1}{\pi} \int_0^{\pi} f(x) \sin nx \, dx = 0.$$

Hence, if $f(x)$ is an even function, $b_n = 0$, $n = 1, 2, 3, \dots$ and we may write its Fourier series expansion as

$$\left. \begin{aligned} f(x) &= \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos nx \\ a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx \end{aligned} \right\} f(x) \text{ even.} \quad (80)$$

Note that half the Fourier coefficients (all the b_n) are zero and that the integration of $f(x) \cos nx$ is carried out over the *half* Fourier interval $(0, \pi)$.

A similar result holds for odd functions. For, if $f(x)$ is odd,

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx \\ &= \frac{1}{\pi} \int_{-\pi}^0 f(x) \sin nx \, dx + \frac{1}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \\ &= \frac{1}{\pi} \int_{\pi}^0 f(-z) \sin n(-z) \, d(-z) + \frac{1}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \\ &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx, \end{aligned}$$

and

$$\begin{aligned}
 a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx \\
 &= \frac{1}{\pi} \int_{-\pi}^0 f(x) \cos nx \, dx + \frac{1}{\pi} \int_0^{\pi} f(x) \cos nx \, dx \\
 &= \frac{1}{\pi} \int_{\pi}^0 f(-x) \cos n(-x) \, d(-x) + \frac{1}{\pi} \int_0^{\pi} f(x) \cos nx \, dx \\
 &= 0.
 \end{aligned}$$

The Fourier series expansion for an odd function is therefore given by

$$\left. \begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin nx \\ b_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx \end{aligned} \right\} f(x) \text{ odd.} \quad (81)$$

In the example of Section 19, the function was odd. Hence all the cosine terms were zero, a fact we verified by actually computing the a_n and b_n . Note that for an odd function no constant term a_0 is present.

23. Examples of expansion of even and odd functions

Let $f(x)$ be a function defined in the full Fourier interval as

$$f(x) = \pi^2 - x^2 \quad -\pi < x < \pi \quad (82)$$

(see Figure 25). Clearly $f(x)$ is an even function, since

$$f(-x) = \pi^2 - (-x)^2 = \pi^2 - x^2 = f(x).$$

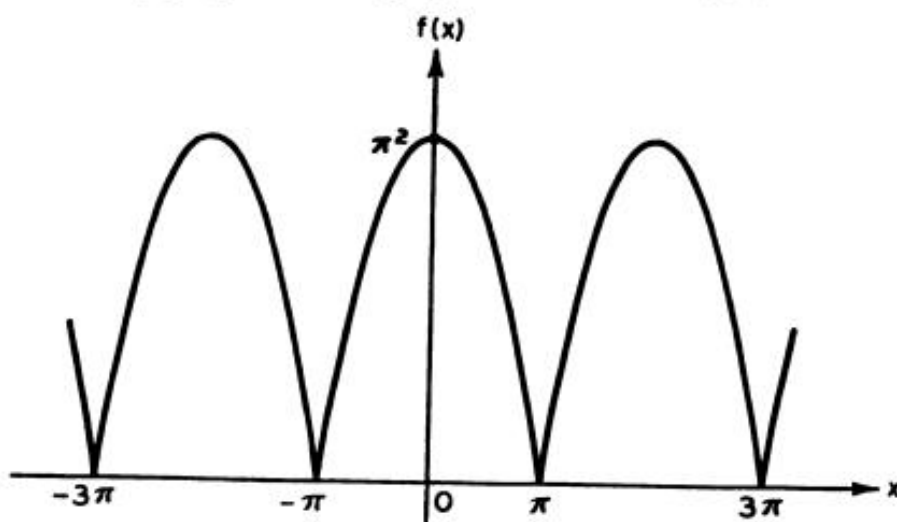


Fig. 25

Hence, by Equations (80),

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^\pi (\pi^2 - x^2) \cos nx \, dx \\ &= \frac{2}{\pi} \left\{ \pi^2 \left(\frac{\sin nx}{n} \right) \Big|_0^\pi - \left[\frac{2x \cos nx}{n^2} + \frac{(\pi^2 x^2 - 2) \sin nx}{n^3} \right] \Big|_0^\pi \right\} \\ &= -\frac{4(-1)^n}{n^3} \quad n \neq 0, \end{aligned}$$

and

$$a_0 = \frac{2}{\pi} \int_0^\pi (\pi^2 - x^2) \, dx = \frac{4}{3} \pi^2.$$

The Fourier series expansion of $f(x)$ is therefore given by

$$f(x) = \frac{2}{3} \pi^2 - 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \cos nx. \quad (83)$$

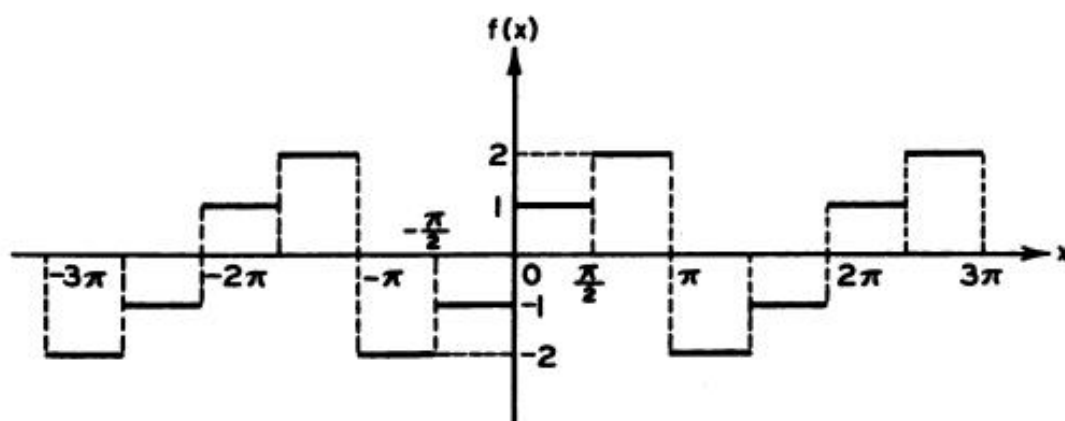


Fig. 26

As an example of an odd function, consider the function $f(x)$ sketched in Figure 26 and analytically defined by

$$\begin{aligned} f(x) &= -2 & -\pi < x < -\frac{\pi}{2}, \\ f(x) &= -1 & -\frac{\pi}{2} < x < 0, \\ f(x) &= +1 & 0 < x < \frac{\pi}{2}, \\ f(x) &= +2 & \frac{\pi}{2} < x < \pi. \end{aligned}$$

By Equations (81) the b_n of $f(x)$ are given by

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi/2} \sin nx \, dx + \frac{2}{\pi} \int_{\pi/2}^{\pi} 2 \sin nx \, dx \\ &= \frac{2}{\pi} \left(-\frac{\cos nx}{n} \right) \Big|_0^{\pi/2} + \frac{4}{\pi} \left(-\frac{\cos nx}{n} \right) \Big|_{\pi/2}^{\pi}, \end{aligned}$$

or

$$\begin{aligned} b_n &= \frac{6}{n\pi} & \text{if } n &= 1, 3, 5, \dots, \\ b_n &= 0 & \text{if } n &= 4, 8, 12, \dots, \\ b_n &= -\frac{4}{n\pi} & \text{if } n &= 2, 6, 10, \dots, \end{aligned}$$

since

$$\cos \frac{n\pi}{2} = 0,$$

if n is odd, and

$$\cos \frac{n\pi}{2} = (-1)^{n/2}$$

if n is even (see Section 28). The Fourier series expansion of $f(x)$ is therefore given by

$$f(x) = \frac{6}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{\sin nx}{n} - \frac{4}{\pi} \sum_{n=2,6,10,\dots}^{\infty} \frac{\sin nx}{n}. \quad (84)$$

24. Half Fourier intervals

In many applications we are interested in a function $f(x)$ defined only in the *half Fourier interval* $(0, \pi)$. To expand such a function all that need be done is to *prolong* it into the other half interval $(-\pi, 0)$ and then expand this function—now defined in the full Fourier interval $(-\pi, \pi)$ —by the usual Fourier series techniques. This is possible because no matter how we extend the function $f(x)$ into the interval $(-\pi, 0)$, it will still represent the desired function in $(0, \pi)$, as well as in $(2\pi, 3\pi)$, $(4\pi, 5\pi)$ etc.

For example, consider the function $f(x) = x$ defined in the half Fourier interval $(0, \pi)$. We may prolong this function to the

half interval $(-\pi, 0)$ in an infinity of ways. In Figure 27a we have extended the function as an *odd* function, in Figure 27b we have extended it as an *even* function, and finally in Figure 27c we have prolonged $f(x)$ to $(-\pi, 0)$ so that it is neither an odd nor an even function. The series expansions of $f(x)$ as given by Figures 27a, 27b, or 27c all represent the function $f(x) = x$ in the

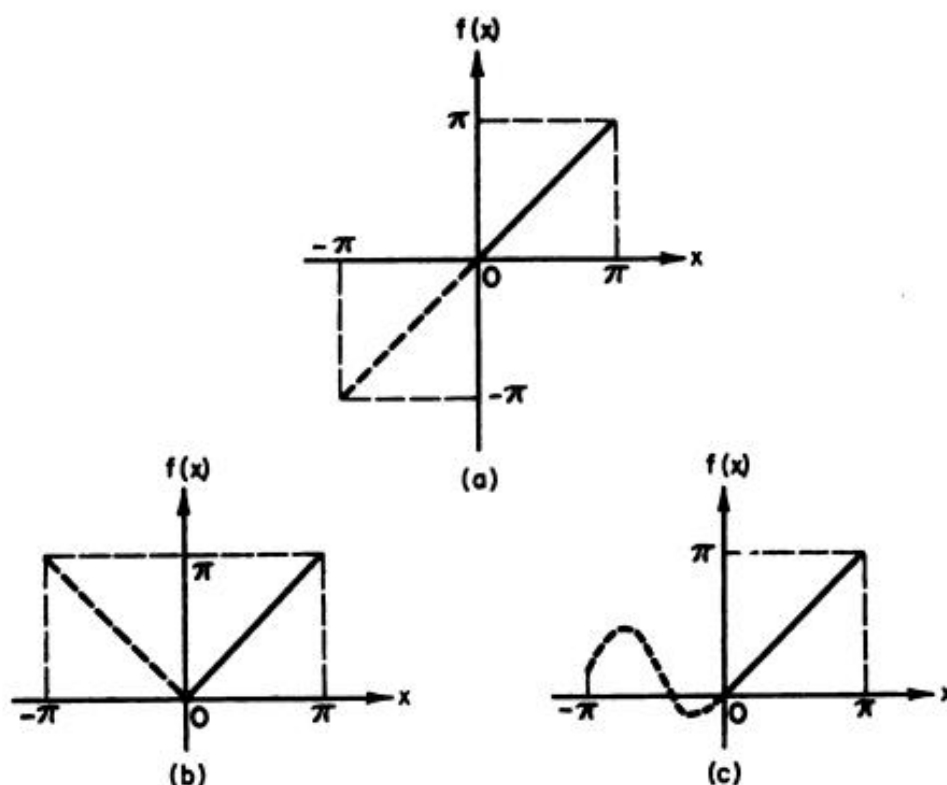


Fig. 27

interval $(0, \pi)$. If we have a choice in the prolongation of a function, it is advisable to prolong it in such a way as to reduce as far as possible the effort of calculating the coefficients of the Fourier series. This would indicate that an even or odd prolongation is preferable, since it eliminates the necessity of calculating half the Fourier coefficients. Sometimes the cosine rather than the sine, expansion, or vice versa, is required by the nature of the problem, as will be seen in the examples of the next chapter. Also, even if we have a free choice of even or odd prolongations, one may be more desirable than the other, as will be illustrated in the next section.

25. Examples of even and odd prolongations

Let us consider the problem of expanding the function $y = x$ given in the half Fourier interval $(0, \pi)$ into a cosine series, that is, of prolonging y as an *even* function (see Figure 27b). In this case we know by Equations (80) that

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^\pi x \cos nx \, dx = \frac{2}{\pi} \left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2} \right] \Big|_0^\pi \\ &= \frac{2}{\pi n^2} [(-1)^n - 1], \quad n \neq 0, \end{aligned}$$

or

$$\begin{aligned} a_n &= 0 & \text{if } n &= 2, 4, 6, \dots, \\ a_n &= -\frac{4}{\pi n^2} & \text{if } n &= 1, 3, 5, \dots, \end{aligned}$$

and, for $n = 0$,

$$a_0 = \frac{2}{\pi} \int_0^\pi x \, dx = \pi.$$

Hence

$$x = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{\cos nx}{n^2}, \quad 0 \leq x < \pi. \quad (85)$$

Let us now expand $f(x) = x$, given in the half Fourier interval $(0, \pi)$ into a sine series (see Figure 27a), that is, let us prolong $f(x)$ as an *odd* function. For this case we know by Equations (81),

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^\pi x \sin nx \, dx = \frac{2}{\pi} \left(-\frac{x \cos nx}{n} + \frac{\sin nx}{n^2} \right) \Big|_0^\pi \\ &= \frac{2}{\pi} \left(-\frac{\pi \cos n\pi}{n} \right) = -\frac{2}{n} (-1)^n, \end{aligned}$$

and hence

$$x = -2 \sum_{n=1}^{\infty} (-1)^n \frac{\sin nx}{n}, \quad -\pi < x < \pi. \quad (86)$$

On comparing this expansion with Equation (85), we see that x as given by the expansion of Equation (85) converges more rapidly than the expansion of Equation (86), since the terms in Equation (85) are decreasing as the *square* of n , and moreover half the

cosine terms are missing. Hence, if we are free to prolong x in the interval $(-\pi, 0)$ as an even or odd function, the even prolongation is to be preferred in this case.

26. Points of discontinuity

The question arises as to what value the series will yield at points where a function is not continuous. It may be proved (see Section 40) that the series yields the *average* of the values at the left and right of the points of discontinuity. Precisely, if $x = \xi$ is a point of discontinuity,

$$f(\xi) = \frac{f(\xi-) + f(\xi+)}{2},$$

where

$$f(\xi-) = \lim_{\substack{x \rightarrow \xi \\ x < \xi}} f(x),$$

and

$$f(\xi+) = \lim_{\substack{x \rightarrow \xi \\ x > \xi}} f(x).$$

In the example of Section 19 (the square wave), the function is discontinuous at the points $x = 0, \pm\pi, \pm2\pi, \dots$, and

$$\begin{aligned} f(n\pi+) &= +h & \text{for } n \text{ even,} \\ f(n\pi+) &= -h & \text{for } n \text{ odd,} \\ f(n\pi-) &= -h & \text{for } n \text{ even,} \\ f(n\pi-) &= +h & \text{for } n \text{ odd.} \end{aligned}$$

Hence the sum of the series at $x = n\pi$ equals

$$f(n\pi) = \frac{\pm h \mp h}{2} = 0.$$

Similarly, in the example of Section 21 (Figure 24),

$$f(1 + 2n\pi) = 1, \quad n = 0, \pm 1, \pm 2, \dots,$$

and in the example of Section 23 (Figure 26),

$$\begin{aligned} f(-\pi) &= 0 & f\left(\frac{\pi}{2}\right) &= \frac{3}{2} \\ f\left(-\frac{\pi}{2}\right) &= -\frac{3}{2} & f(\pi) &= 0 \\ f(0) &= 0 & \dots & \dots \dots \end{aligned}$$

27. Trigonometric approximations

Let $f(x)$ be a periodic function with period 2π . The question sometime arises as to what is the "best approximation" to $f(x)$ obtainable by using only a *finite* number of sines and cosines. By "best approximation" to $f(x)$ we usually mean a function $A(x)$ which makes the integral

$$I = \int_{-\pi}^{\pi} [f(x) - A(x)]^2 dx \quad (87)$$

as small as possible. The integral I is called "the mean square error" and $A(x)$ approximates $f(x)$ "in the mean square sense."

It will now be shown that the best approximation $A(x)$ to $f(x)$ is one in which the coefficients of the trigonometric terms are precisely the Fourier coefficients of $f(x)$. Toward this end consider the finite series

$$A(x) = \frac{a_0}{2} + \sum_{n=1}^N (a_n \cos nx + b_n \sin nx) \quad (88)$$

in which the a_n and b_n are to be determined such that

$$\int_{-\pi}^{\pi} [f(x) - A(x)]^2 dx$$

shall be a minimum. Denote the above integral by

$$I(a_0, a_1, \dots, a_N, b_1, b_2, \dots, b_N),$$

that is, I is a function of the undetermined coefficients a_n and b_n only. By the calculus the derivatives of I with respect to the a_n and b_n must equal zero for I to be a minimum. Hence

$$\begin{aligned} 0 &= \frac{\partial I}{\partial a_k} = 2 \int_{-\pi}^{\pi} [f(x) - A(x)] \frac{\partial A}{\partial a_k} dx \\ &= 2 \int_{-\pi}^{\pi} [f(x) - A(x)] \cos kx dx, \end{aligned}$$

or

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos kx dx &= \int_{-\pi}^{\pi} \left(\frac{a_0}{2} \cos kx \right) dx \\ &+ \sum_{n=1}^N \int_{-\pi}^{\pi} (a_n \cos nx + b_n \sin nx) \cos kx dx. \end{aligned}$$

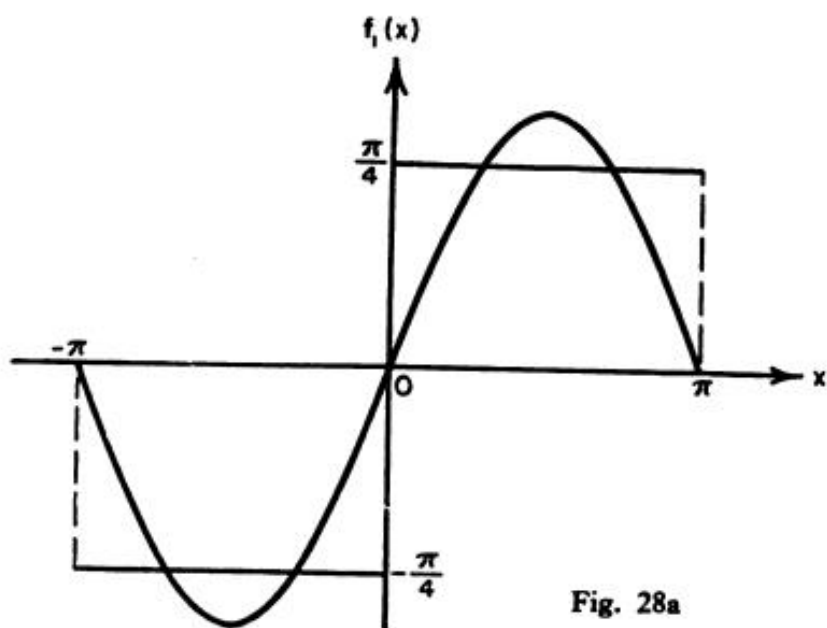


Fig. 28a

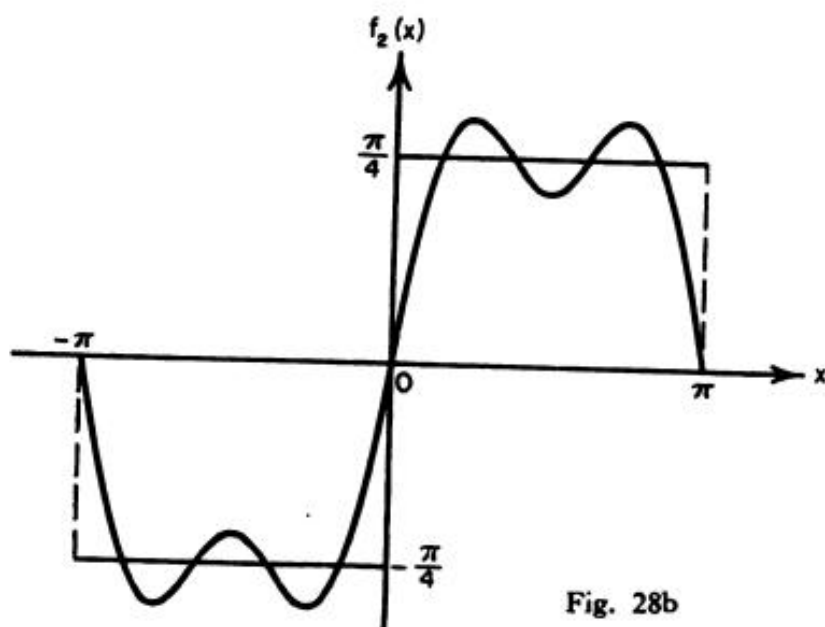


Fig. 28b

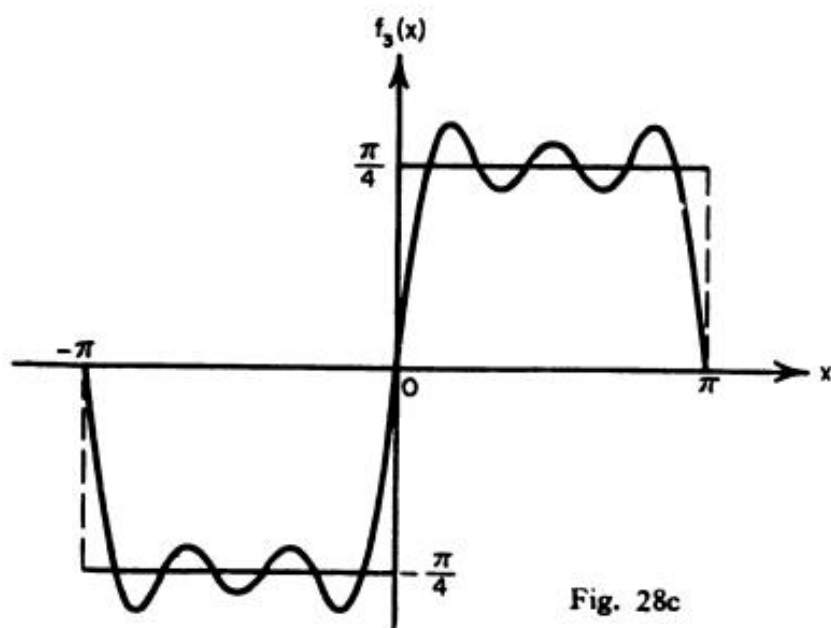


Fig. 28c

By the orthogonality conditions of Equations (68) the term on the right is πa_k ; ($k = 0, 1, 2, \dots, N$) and hence

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx \, dx, \quad k = 0, 1, \dots, N.$$

In a similar fashion, on differentiating I with respect to b_k and setting $\partial I / \partial b_k$ equal to zero we obtain

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx \, dx, \quad k = 1, 2, \dots, N.$$

It is thus seen that the a_k and b_k of Equation (88) are precisely the Fourier coefficients. For example, the best approximation to the square wave (with $h = \pi/4$) of Section 19, using one term, is

$$f_1(x) = \sin x.$$

Using two terms, the best approximation is

$$f_2(x) = \sin x + \frac{1}{3} \sin 3x,$$

and using N terms, the best approximation is

$$f_N(x) = \sum_{n=1,3,5,\dots}^{2N-1} \frac{\sin nx}{n}.$$

The graphs of the functions $f_1(x)$, $f_2(x)$, $f_3(x)$ are plotted in Figure 28.

28. Trigonometric identities

In the Fourier series expansions of the previous sections we made use of certain trigonometric identities like $\cos n\pi = (-1)^n$ for $n = 0, 1, 2, \dots$ and $\cos (n\pi/2) = (-1)^{n/2}$ if n is even. It is convenient to have for reference the following table of similar formulas.

	n	n even	n odd	$n/2$ odd	$n/2$ even
$\sin n\pi$	0	0	0	0	0
$\cos n\pi$	$(-1)^n$	+ 1	- 1	+ 1	+ 1
$\sin \frac{n\pi}{2}$		0	$(-1)^{(n-1)/2}$	0	0
$\cos \frac{n\pi}{2}$		$(-1)^{n/2}$	0	- 1	+ 1
$\sin \frac{n\pi}{4}$			$\frac{\sqrt{2}}{2}(-1)^{(n^2+4n+11)/8}$	$(-1)^{(n-2)/4}$	0

29. Fourier series expansion in $(-L, L)$

If $f(x)$ is a periodic function of period 2π we have seen that it has the Fourier series expansion

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

In many engineering problems, the interval of interest is not restricted to 2π but may be of arbitrary length, say $2L$. To obtain a Fourier series expansion for a function $f(x)$, which is periodic of period $2L$ and is given in the fundamental interval $(-L, L)$ we introduce the change of variable

$$z = \frac{\pi}{L} x \tag{89}$$

and let

$$f(x) = f\left(\frac{Lz}{\pi}\right) = \phi(z).$$

Here $\phi(z)$ is a new function of the variable z which has period 2π , for we see by Equation (89) that when $x = -L$, $z = -\pi$, and when $x = L$, $z = \pi$. Since $\phi(z)$ is periodic of period 2π we can expand it as was done in previous sections, obtaining

$$\phi(z) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nz + b_n \sin nz),$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(z) \cos nz \, dz,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \phi(z) \sin nz \, dz.$$

If we now return to the original variable x , we have

$$\phi(z) = f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x \right) \tag{90}$$

where

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-L}^L f(x) \cos \frac{n\pi}{L}x \, d\left(\frac{\pi}{L}x\right) = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi}{L}x \, dx, \\ b_n &= \frac{1}{\pi} \int_{-L}^L f(x) \sin \frac{n\pi}{L}x \, d\left(\frac{\pi}{L}x\right) = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi}{L}x \, dx. \end{aligned} \quad (91)$$

We have thus obtained the expansion for a function $f(x)$ of arbitrary period $2L$, and as mentioned in Section 20, we may also write

$$\begin{aligned} a_n &= \frac{1}{L} \int_{\alpha}^{\alpha+2L} f(x) \cos \frac{n\pi}{L}x \, dx, \\ b_n &= \frac{1}{L} \int_{\alpha}^{\alpha+2L} f(x) \sin \frac{n\pi}{L}x \, dx, \end{aligned}$$

where α is any real number. A commonly used value of α is zero. In this case, the Fourier coefficients become

$$\begin{aligned} a_n &= \frac{1}{L} \int_0^{2L} f(x) \cos \frac{n\pi}{L}x \, dx, \\ b_n &= \frac{1}{L} \int_0^{2L} f(x) \sin \frac{n\pi}{L}x \, dx. \end{aligned} \quad (92)$$

30. Example of a Fourier series in $(-L, L)$

As an example illustrating the formulas of the previous section, consider the function $f(x)$ defined in the interval $(-1, 1)$ as

$$f(x) = x^3 \quad -1 < x < 1.$$

If we repeat this function periodically assuming $(-1, 1)$ to be the full Fourier interval, we obtain the function sketched in Figure 29. Since $2L = 1 - (-1) = 2$, $L = 1$, we have by Equations (91),

$$a_n = \int_{-1}^1 x^3 \cos n\pi x \, dx, \quad b_n = \int_{-1}^1 x^3 \sin n\pi x \, dx.$$

We readily compute the a_n and b_n :

$$a_n = \frac{1}{n^4\pi^4} [(3n^2\pi^2x^2 - 6) \cos n\pi x + (n^3\pi^3x^3 - 6n\pi x) \sin n\pi x] \Big|_{-1}^{+1} \\ = 0 \quad n = 1, 2, \dots,$$

$$a_0 = \int_{-1}^1 x^3 dx = 0,$$

$$b_n = \frac{1}{n^4\pi^4} [(3n^2\pi^2x^2 - 6) \sin n\pi x - (n^3\pi^3x^3 - 6n\pi x) \cos n\pi x] \Big|_{-1}^{+1} \\ = \frac{2(6 - n^2\pi^2)}{n^3\pi^3} (-1)^n.$$

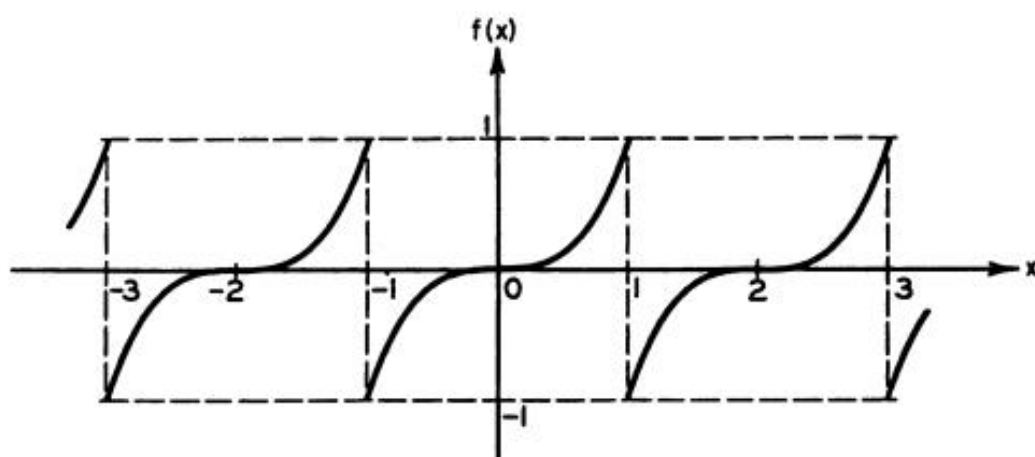


Fig. 29

Hence, by Equation (90),

$$f(x) = \frac{2}{\pi^3} \sum_{n=1}^{\infty} \frac{(6 - n^2\pi^2)}{n^3} (-1)^n \sin n\pi x.$$

31. Even and odd functions in $(-L, L)$

If $f(x)$ is an even function of period $2L$, we can write for it an expansion analogous to that of Equation (80) by the techniques used in Section 22. We leave it to the reader to verify that in this case

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L}x$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi}{L} x dx, \quad n = 0, 1, 2, \dots \quad (93)$$

Correspondingly if $f(x)$ is an odd function of period $2L$, the expansion analogous to that of Equations (81) becomes

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \quad (94)$$

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx, \quad n = 1, 2, 3, \dots$$

32. Examples of even and odd functions in $(-L, L)$

Consider the odd periodic function $f(x)$

$$f(x) = \frac{x}{2} \quad -4 < x < 4$$

illustrated in Figure 30. Since the full Fourier interval $2L$ is 8, its Fourier coefficients are given by Equation (94) as

$$\begin{aligned} b_n &= \frac{1}{2} \int_0^4 \frac{x}{2} \sin \frac{n\pi}{4} x dx \\ &= \left(-\frac{x}{n\pi} \cos \frac{n\pi}{4} x + \frac{4}{n^2\pi^2} \sin \frac{n\pi}{4} x \right) \Big|_0^4 \\ &= -\frac{4}{n\pi} (-1)^n, \end{aligned}$$

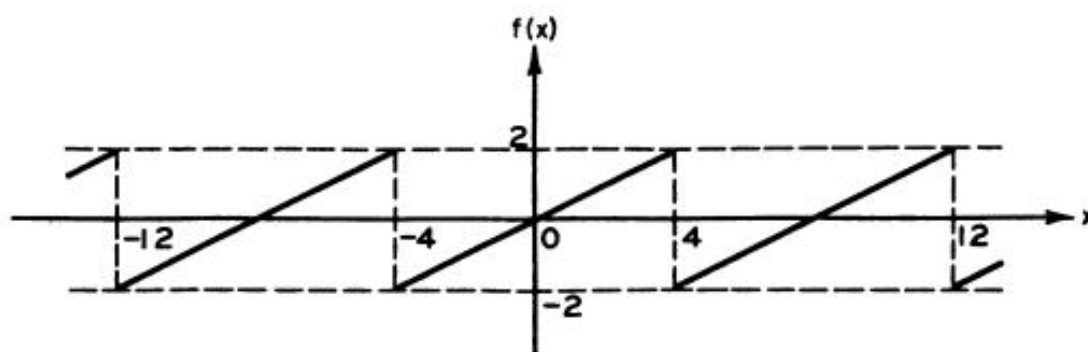


Fig. 30

and

$$f(x) = -\frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (-1)^n \sin \frac{n\pi}{4} x. \quad (95)$$

As an example of an even function consider the function illustrated in Figure 31, whose period $2L = 8$. By Equations (93) its Fourier coefficients are given by

$$\begin{aligned} a_n &= \frac{2}{4} \int_0^1 \cos \frac{n\pi}{4} x \, dx - \frac{2}{4} \int_3^4 \cos \frac{n\pi}{4} x \, dx \\ &= \frac{2}{n\pi} \left(\sin \frac{n\pi}{4} + \sin \frac{3n\pi}{4} \right) \\ &= \frac{2}{n\pi} \sin \frac{n\pi}{4} [1 + (-1)^{n+1}], \quad n \neq 0. \end{aligned}$$

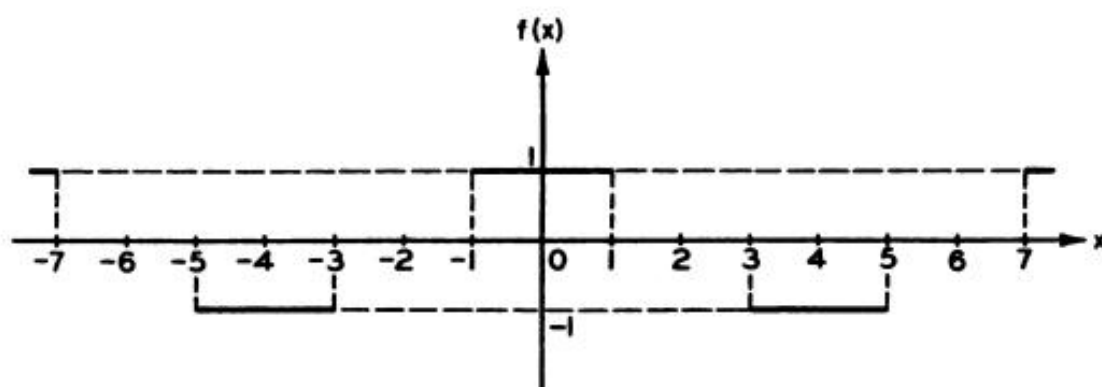


Fig. 31

Hence

$$\begin{aligned} a_n &= 0 && \text{if } n \text{ is even,} \\ a_n &= \frac{4}{n\pi} \frac{\sqrt{2}}{2} (-1)^{(n^2+4n+11)/8} && \text{if } n \text{ is odd.} \end{aligned}$$

(See Section 28). Also

$$a_0 = \frac{1}{2} \int_0^1 dx - \frac{1}{2} \int_3^4 dx = 0.$$

The Fourier series expansion of the function of Figure 31 is thus

$$f(x) = \frac{2\sqrt{2}}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n} (-1)^{(n^2+4n+11)/8} \cos \frac{n\pi}{4} x. \quad (96)$$

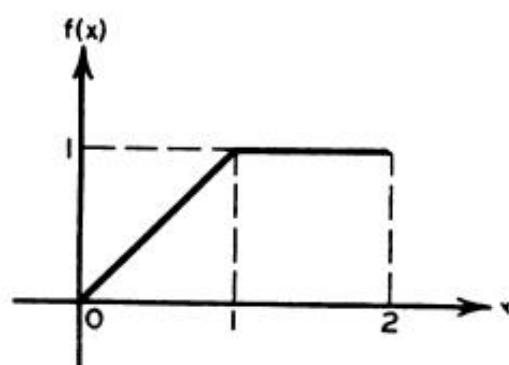
33. Prolongation of functions in $(-L, L)$ 

Fig. 32

If $f(x)$ is given in the half Fourier interval $(0, L)$, its expansion may be obtained by prolonging it into $(-L, 0)$ as an even function, an odd function, or neither (see Section 24). Consider, for example, the function $f(x)$,

$$\begin{aligned} f(x) &= x & 0 \leq x \leq 1, \\ f(x) &= 1 & 1 \leq x \leq 2, \end{aligned}$$

shown in Figure 32, which is defined in the half Fourier interval $(0, 2)$. If we extend this function as an *odd* function, it will appear as in Figure 33. Its Fourier coefficients are given by

$$\begin{aligned} b_n &= \int_0^1 x \sin \frac{n\pi}{2}x \, dx + \int_1^2 \sin \frac{n\pi}{2}x \, dx \\ &= \left(-\frac{2x}{n\pi} \cos \frac{n\pi}{2}x + \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2}x \right) \Big|_0^1 - \left(\frac{2}{n\pi} \cos \frac{n\pi}{2}x \right) \Big|_1^2, \\ b_n &= -\frac{2}{n\pi} (-1)^n & \text{if } n \text{ is even,} \\ b_n &= \frac{4}{n^2\pi^2} (-1)^{(n-1)/2} - \frac{2}{n\pi} (-1)^n & \text{if } n \text{ is odd.} \end{aligned}$$

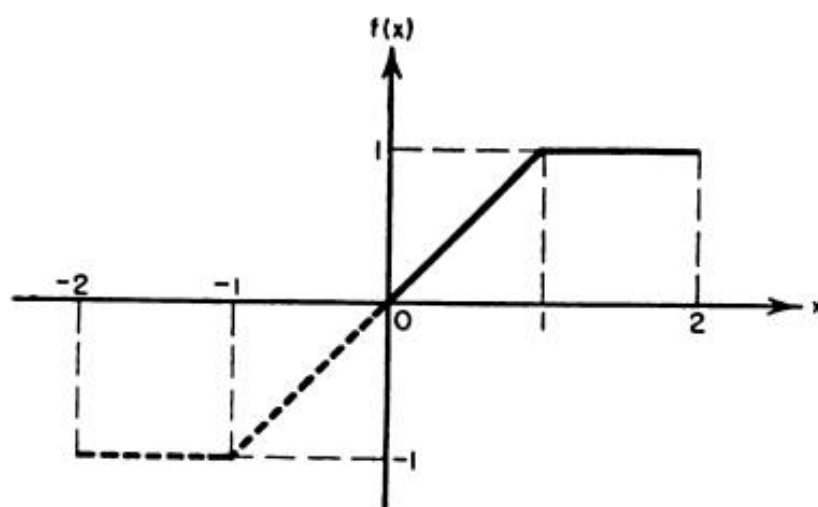


Fig. 33

Therefore

$$\begin{aligned}
 f(x) &= - \sum_{n=2,4,6,\dots}^{\infty} \frac{2}{n\pi} (-1)^n \sin \frac{n\pi}{2}x \\
 &\quad + \sum_{n=1,3,5,\dots}^{\infty} \frac{4}{n^2\pi^2} (-1)^{(n-1)/2} \sin \frac{n\pi}{2}x \\
 &\quad - \sum_{n=1,3,5,\dots}^{\infty} \frac{2}{n\pi} (-1)^n \sin \frac{n\pi}{2}x \\
 &= \frac{4}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} (-1)^{(n-1)/2} \sin \frac{n\pi}{2}x \\
 &\quad - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (-1)^n \sin \frac{n\pi}{2}x. \tag{97}
 \end{aligned}$$

[We note, of course, that

$$\frac{2}{n\pi} (-1)^n = \frac{2}{n\pi}$$

if n is even, but we prefer to keep the $(-1)^n$ factor so that the first and third series may be readily combined.]

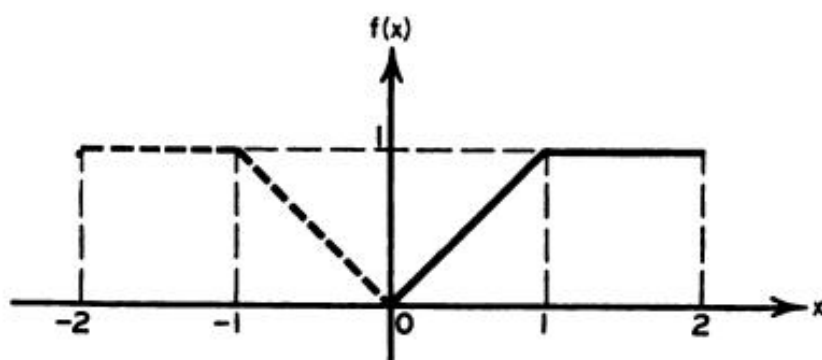


Fig. 34

The function defined by Figure 32 could also be extended into $(-L, 0)$ as an even function. In this case, it would appear as illustrated in Figure 34 above. The reader may obtain the corresponding cosine series expansion for this function (see Problem 19).

34. Phase angles

An alternative form of Fourier series may be considered when both sine and cosine terms are present. Suppose we have a function $f(x)$ of period $2L$ given in the interval $(-L, L)$, whose Fourier series expansion is given by

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x \right), \quad (98)$$

where

$$a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi}{L}x \, dx,$$

$$b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi}{L}x \, dx.$$

The sum of two corresponding terms of Equation (98),

$$a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x, \quad (99)$$

is also a sinusoidal function and may therefore be written

$$a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x = c_n \sin \left(\frac{n\pi}{L}x + \phi_n \right). \quad (100)$$

To determine c_n and ϕ_n , expand the right-hand side of the above equation :

$$c_n \sin \left(\frac{n\pi}{L}x + \phi_n \right) = c_n \sin \frac{n\pi}{L}x \cos \phi_n + c_n \sin \phi_n \cos \frac{n\pi}{L}x,$$

and equate coefficients of

$$\sin \frac{n\pi}{L}x \quad \text{and} \quad \cos \frac{n\pi}{L}x$$

on both sides :

$$a_n = c_n \sin \phi_n, \quad b_n = c_n \cos \phi_n. \quad (101)$$

Hence on squaring and adding,

$$c_n = \sqrt{a_n^2 + b_n^2}, \quad (102)$$

while taking the ratio,

$$\phi_n = \arctan \left(\frac{a_n}{b_n} \right). \quad (103)$$

The Fourier expansion of Equation (98) thus becomes

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} c_n \sin \left(\frac{n\pi}{L}x + \phi_n \right). \quad (104)$$

We could also write

$$a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x = c_n \cos \left(\frac{n\pi}{L}x + \phi_n \right),$$

and we determine as above that

$$a_n = c_n \cos \phi_n, \quad b_n = -c_n \sin \phi_n, \quad (105)$$

and hence

$$c_n = \sqrt{a_n^2 + b_n^2}, \quad \phi_n = \arctan \left(-\frac{b_n}{a_n} \right). \quad (106)$$

Our Fourier series expansion therefore becomes

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} c_n \cos \left(\frac{n\pi}{L}x + \phi_n \right). \quad (107)$$

Care must be exercised in computing ϕ_n from Equation (103) [or (106)] since there are two possible determinations of ϕ_n which satisfy the equation. The value which must be selected is the one which satisfies both of Equations (101) [or (105)].

35. Example of a Fourier series involving a phase angle

Consider the function of Figure 24 (Section 21). Its Fourier series expansion was determined to be [Equation (77)]

$$f(x) = 1 + \frac{2}{\pi} \sum_{n=1}^{\infty} \left(+\frac{\sin n}{n} \cos nx - \frac{\cos n}{n} \sin nx \right). \quad (108)$$

To write this series in terms of a phase angle [Equation (104)] we have first from Equation (102) that

$$c_n = \sqrt{\left(+\frac{\sin n}{n} \right)^2 + \left(-\frac{\cos n}{n} \right)^2} = \frac{1}{n},$$

and from Equation (103)

$$\phi_n = \arctan \left(-\frac{\sin n}{\cos n} \right) = \arctan (-\tan n) = \pi - n.$$

The phase angle form of Equation (108) is thus [from Equation (104)]

$$\begin{aligned} f(x) &= 1 + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin (nx + \pi - n) \\ &= 1 - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin n(x-1). \end{aligned} \quad (109)$$

36. Complex Fourier series

In many engineering applications of Fourier series it is more convenient to expand a function into series of *exponentials* rather than into series involving sines and/or cosines. Since the trigonometric and exponential functions are intimately connected by Euler's formulas ([24], Section 5.12),

$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \quad \sin \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

we can use these relations in Equation (90) to obtain the *complex Fourier series* expansion of the function $f(x)$. If we make this substitution and for simplicity let

$$\omega_n = \frac{n\pi}{L}, \quad n = 0, \pm 1, \pm 2, \dots, \quad (110)$$

the Fourier series expansion of Equation (90) becomes

$$\begin{aligned} f(x) &= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \left(\frac{e^{j\omega_n x} + e^{-j\omega_n x}}{2} \right) + \sum_{n=1}^{\infty} b_n \left(\frac{e^{j\omega_n x} - e^{-j\omega_n x}}{2j} \right) \\ &= \frac{1}{2}a_0 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n - jb_n) e^{j\omega_n x} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n + jb_n) e^{-j\omega_n x}. \end{aligned} \quad (111)$$

Letting

$$c_n = a_n - jb_n \quad (112)$$

and with the Fourier coefficients a_n and b_n of Equation (91), we have

$$\begin{aligned} c_n = a_n - jb_n &= \frac{1}{L} \int_{-L}^L f(x) (\cos \omega_n x - j \sin \omega_n x) dx \\ &= \frac{1}{L} \int_{-L}^L f(x) e^{-j\omega_n x} dx. \end{aligned} \quad (113)$$

Similarly, $a_n + jb_n = \bar{c}_n$, the complex conjugate of c_n ,

$$\bar{c}_n = a_n + jb_n = \frac{1}{L} \int_{-L}^L f(x) e^{j\omega_n x} dx.$$

But from Equation (110),

$$\omega_{-n} = \frac{(-n)\pi}{L} = -\frac{n\pi}{L} = -\omega_n,$$

and if we replace n by $-n$ in Equation (113),

$$c_{-n} = \frac{1}{L} \int_{-L}^L f(x) e^{-j\omega_n x} dx = \frac{1}{L} \int_{-L}^L f(x) e^{j\omega_n x} dx = \bar{c}_n. \quad (114)$$

Hence

$$\begin{aligned} f(x) &= \frac{1}{2}a_0 + \frac{1}{2} \sum_{n=1}^{\infty} c_n e^{j\omega_n x} + \frac{1}{2} \sum_{n=1}^{\infty} \bar{c}_n e^{-j\omega_n x} \\ &= \frac{1}{2} \left(a_0 + \sum_{n=1}^{\infty} c_n e^{j\omega_n x} + \sum_{n=1}^{\infty} c_{-n} e^{-j\omega_n x} \right). \end{aligned} \quad (115)$$

Let $n = -m$ in the second summation on the right; then

$$\begin{aligned} \sum_{n=1}^{\infty} c_{-n} e^{-j\omega_n x} &= \sum_{m=-1}^{-\infty} c_m e^{-j\omega_{-m} x} \\ &= \sum_{m=-\infty}^{-1} c_m e^{j\omega_m x} = \sum_{n=-\infty}^{-1} c_n e^{j\omega_n x}. \end{aligned}$$

Thus Equation (115) may be written

$$f(x) = \frac{1}{2} \left(a_0 + \sum_{n=1}^{\infty} c_n e^{j\omega_n x} + \sum_{n=-\infty}^{-1} c_n e^{j\omega_n x} \right).$$

But

$$a_0 = \frac{1}{L} \int_{-L}^L f(x) dx = c_0 = c_0 e^{j\omega_0 x},$$

since

$$\omega_0 = \frac{0\pi}{L} = 0,$$

so that finally

$$f(x) = \frac{1}{2} \sum_{n=-\infty}^{+\infty} c_n e^{j\omega_n x},$$

where (116)

$$c_n = \frac{1}{L} \int_{-L}^L f(x) e^{-j\omega_n x} dx, \quad n = 0, \pm 1, \pm 2, \dots$$

Equations (116) are known as the complex Fourier series expansion (or exponential Fourier series) of $f(x)$. The set of coefficients $\{c_n\}$ is often called the Fourier spectrum or simply the *spectrum* of $f(x)$.

The complex form of the Fourier series is much more compact than the trigonometric form. It is useful, for example, in deriving the Fourier integral (see Chapter IV). One disadvantage of the exponential form is that it gives no simplification in the case of even and odd functions.

37. Example of a complex Fourier series

Let us expand the function $f(x) = e^x$ given in the full Fourier interval $(-1, 1)$ (Figure 35) into a complex Fourier series. With

$$L = 1, \quad \omega_n = \frac{n\pi}{L} = n\pi,$$

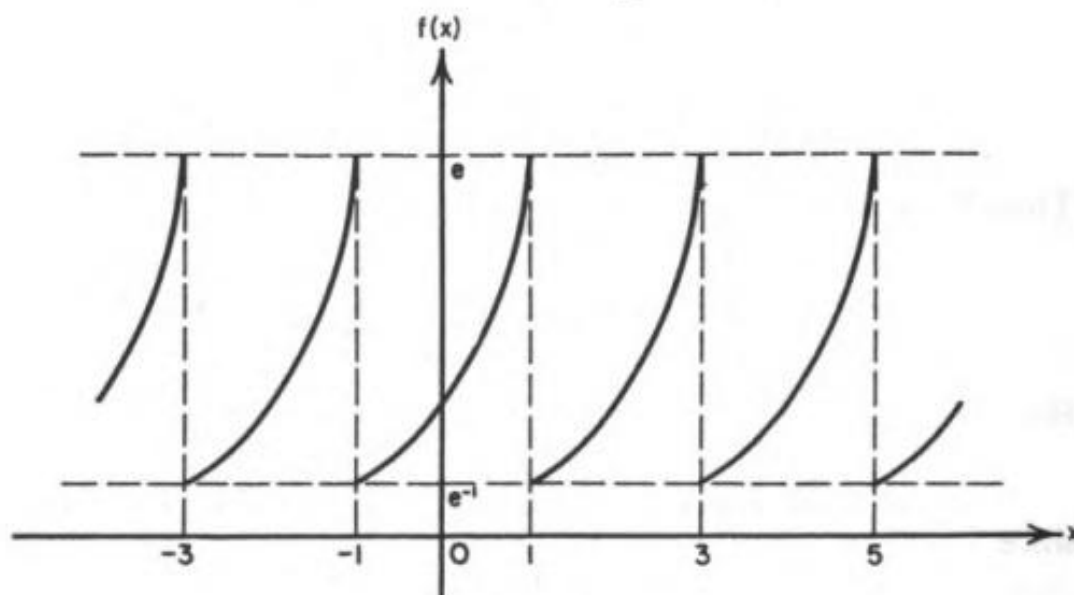


Fig. 35

Equations (116) give

$$\begin{aligned} c_n &= \int_{-1}^1 e^x e^{-jn\pi x} dx = \frac{e^{x(1-jn\pi)}}{1-jn\pi} \Big|_{-1}^1 \\ &= \frac{e^{(1-jn\pi)} - e^{-(1-jn\pi)}}{1-jn\pi}. \end{aligned}$$

But

$$e^{jn\pi} = \cos n\pi + j \sin n\pi = (-1)^n$$

and

$$c_n = \frac{(-1)^n (e - e^{-1})}{1 - jn\pi} = \frac{(-1)^n (1 + jn\pi) (e - e^{-1})}{1 + n^2\pi^2}.$$

Since

$$\frac{e - e^{-1}}{2} = 1.1752,$$

the complex Fourier series of e^x becomes, by Equation (116),

$$f(x) = 1.1752 \sum_{n=-\infty}^{\infty} \frac{(-1)^n (1 + jn\pi)}{1 + n^2\pi^2} e^{jn\pi x}. \quad (117)$$

38. Double trigonometric series

In many applications it is found necessary to expand a function of *two* variables into a trigonometric series. Consider a function $f(x, y)$ periodic of period 2π in *both* x and y . Then, if we hold y fixed, we can expand $f(x, y)$ into an ordinary Fourier series, in which, however, the coefficients will be functions of y , viz.,

$$f(x, y) = \sum_{n=0}^{\infty} [A_n(y) \cos nx + B_n(y) \sin nx].$$

If we assume that the functions $A_n(y)$ and $B_n(y)$ can be expanded in a Fourier series,

$$A_n(y) = \sum_{m=0}^{\infty} (\alpha_{nm} \cos my + \beta_{nm} \sin my),$$

$$B_n(y) = \sum_{m=0}^{\infty} (\gamma_{nm} \cos my + \eta_{nm} \sin my),$$

and substitute these coefficients in the expression for $f(x,y)$ we obtain terms of the form

$$\begin{array}{ll} \cos nx \cos my, & \cos nx \sin my, \\ \sin nx \cos my, & \sin nx \sin my, \end{array}$$

with *constant* coefficients. Hence, in general we may write

$$f(x,y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} (a_{nm} \cos nx \cos my + b_{nm} \cos nx \sin my + c_{nm} \sin nx \cos my + d_{nm} \sin nx \sin my). \quad (118)$$

In order to calculate the coefficients a_{nm} , b_{nm} , c_{nm} , and d_{nm} we use the integration procedure employed in ordinary Fourier series, but we apply it *twice*. Multiplying each side of the above equation by $\sin \alpha x$, where α is a nonzero integer, and integrating with respect to x from $-\pi$ to π , we obtain, because of the orthogonality conditions,

$$\int_{-\pi}^{\pi} f(x,y) \sin \alpha x \, dx = \pi \sum_{m=0}^{\infty} c_{\alpha m} \cos my + \pi \sum_{m=0}^{\infty} d_{\alpha m} \sin my. \quad (119)$$

Multiplying this equation by $\sin \beta y$ and integrating with respect to y from $-\pi$ to π , we obtain again, by the orthogonality conditions,

$$\int_{-\pi}^{\pi} \left[\int_{-\pi}^{\pi} f(x,y) \sin \alpha x \, dx \right] \sin \beta y \, dy = \pi^2 d_{\alpha \beta}$$

or

$$d_{\alpha \beta} = \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) \sin \alpha x \sin \beta y \, dx \, dy. \quad (120)$$

In a similar fashion,

$$\left. \begin{aligned} a_{\alpha \beta} &= \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) \cos \alpha x \cos \beta y \, dx \, dy \\ b_{\alpha \beta} &= \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) \cos \alpha x \sin \beta y \, dx \, dy \\ c_{\alpha \beta} &= \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) \sin \alpha x \cos \beta y \, dx \, dy. \end{aligned} \right\} \quad (121)$$

Equations (120) and (121) are valid for $\alpha, \beta = 1, 2, 3, \dots$. The remaining constants a_{0m} , a_{n0} , b_{0m} , b_{n0} , c_{0m} , c_{n0} , d_{0m} , d_{n0} are co-

efficients of terms involving only *one* variable. We see immediately that of these constants

$$d_{n0} = d_{0m} = c_{0m} = b_{n0} = 0 \quad (122)$$

since all these terms involve the sine of zero. To evaluate c_{n0} , multiply Equation (119) by $\cos 0y = 1$ and integrate from $-\pi$ to π with respect to y :

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) \sin \alpha x \, dx \, dy = 2\pi^2 c_{\alpha 0},$$

from which

$$c_{\alpha 0} = \frac{1}{2\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) \sin \alpha x \, dx \, dy. \quad (123)$$

In a similar fashion,

$$b_{0\beta} = \frac{1}{2\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) \sin \beta y \, dx \, dy. \quad (124)$$

To calculate a_{n0} , $n \neq 0$, multiply Equation (118) by $\cos \alpha x$ and integrate from $-\pi$ to π with respect to x :

$$\int_{-\pi}^{\pi} f(x,y) \cos \alpha x \, dx = \pi \sum_{m=0}^{\infty} a_{\alpha m} \cos my + \pi \sum_{n=0}^{\infty} b_{\alpha n} \sin ny,$$

and then multiply by $\cos 0y = 1$ and integrate from $-\pi$ to π with respect to y .

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) \cos \alpha x \, dx \, dy = 2\pi^2 a_{\alpha 0},$$

from which

$$a_{\alpha 0} = \frac{1}{2\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) \cos \alpha x \, dx \, dy. \quad (125)$$

In a similar fashion we verify that

$$a_{0\beta} = \frac{1}{2\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) \cos \beta y \, dx \, dy. \quad (126)$$

Finally, a_{00} is calculated by integrating both sides of Equation (118) with respect to both x and y :

$$\int_{-\pi}^{\pi} f(x,y) \, dx = 2\pi \sum_{n=0}^{\infty} a_{0n} \cos ny + 2\pi \sum_{n=0}^{\infty} b_{0n} \sin ny,$$

$$\int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) \, dx \, dy = 4\pi^2 a_{00},$$

from which

$$a_{00} = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) dx dy. \quad (127)$$

It is not our intention to study multiple Fourier series in as exhaustive a fashion as we did Fourier series in one variable. Hence for the present we shall consider only a special case, namely when $f(x,y)$ is an odd function of both x and y . In this case the coefficients

$$a_{00}, a_{0m}, a_{n0}, a_{nm}, b_{0m}, b_{nm}, c_{n0}, c_{nm}$$

vanish, and the double Fourier series reduces to

$$\begin{aligned} f(x,y) &= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} d_{nm} \sin nx \sin my \\ d_{nm} &= \frac{1}{\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(x,y) \sin nx \sin my dx dy \\ &= \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} f(x,y) \sin nx \sin my dx dy. \end{aligned} \quad (128)$$

39. Example of a double Fourier series

Let $f(x,y) = k$ (a constant) in the rectangle $0 < x < \pi$, $0 < y < \pi$ and be an *odd* function of both x and y . The function $f(x,y)$ is sketched in Figure 36 and is defined analytically by the equations

$$\begin{aligned} f(x,y) &= k & 0 < x < \pi, & & 0 < y < \pi, \\ f(x,y) &= k & -\pi < x < 0, & & -\pi < y < 0, \\ f(x,y) &= -k & 0 < x < \pi, & & -\pi < y < 0, \\ f(x,y) &= -k & -\pi < x < 0, & & 0 < y < \pi. \end{aligned}$$

Then by Equations (128),

$$\begin{aligned} d_{nm} &= \frac{4k}{\pi^2} \int_0^{\pi} \int_0^{\pi} \sin nx \sin my dx dy \\ &= \frac{16k}{\pi^2 nm} & \text{if } n \text{ and } m \text{ are odd,} \\ &= 0 & \text{otherwise,} \end{aligned}$$

and the double Fourier series of $f(x,y) = k$ is

$$f(x,y) = k = \frac{16k}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{nm} \sin nx \sin my.$$

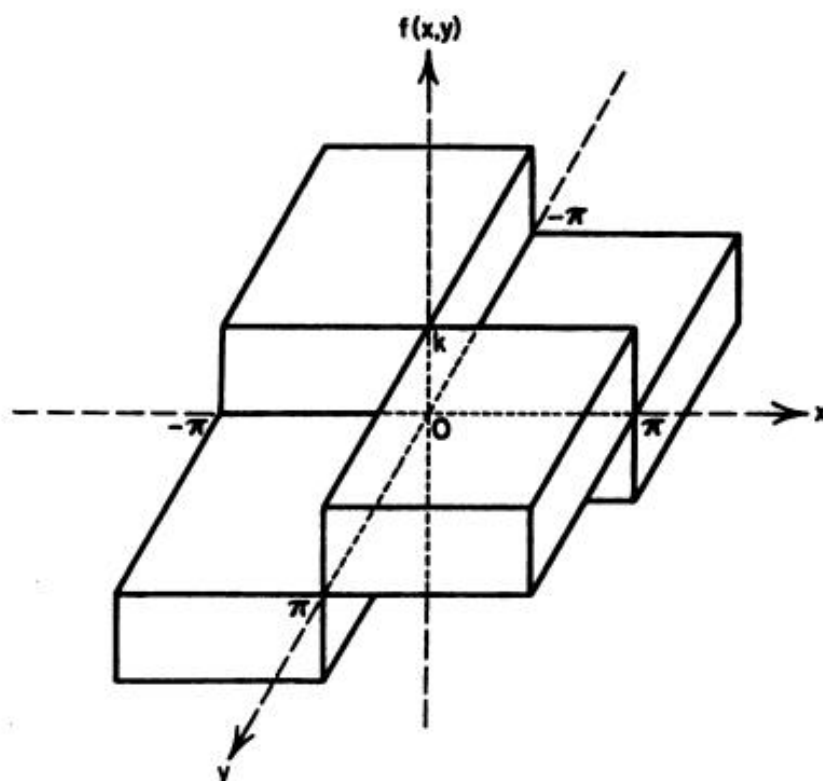


Fig. 36

40. Mathematical conditions on Fourier series expansions

Not all periodic functions can be expanded into a Fourier series, since certain mathematical conditions must be satisfied to guarantee that the sum of the series equals the function. The classical sufficient conditions for a function to be Fourier expandable are known as the *Dirichlet conditions* and are listed here in a form general enough for most engineering purposes.

If $f(x)$ is a single-valued function of x , periodic of period 2π such that in the interval $(-\pi, \pi)$,

- (i) $f(x)$ never becomes infinite,
- (ii) $f(x)$ has at most a finite number of maxima and minima,
- (iii) $f(x)$ has at most a finite number of discontinuities, then $f(x)$ is Fourier expandable.

Certain conditions, slightly weaker than the Dirichlet conditions lead to the following precise theorem whose proof may be found in [6].

Theorem. Let $f(x)$ be a periodic function of period 2π . Suppose that the interval $(-\pi, \pi)$ can be broken up into a finite number of subintervals in each of which $f(x)$ is continuous. Then the Fourier series for $f(x)$,

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx, \quad n = 0, 1, 2, \dots,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx, \quad n = 1, 2, \dots,$$

converges to the value

$$\frac{1}{2}[f(x+) + f(x-)]$$

at every point at which $f(x)$ has a right- and left-hand derivative.

If x is a point of continuity, then $f(x+) = f(x-) = f(x)$, and the series converges to $f(x)$.

In summary, we may say that while not all functions possess Fourier series expansions, almost any function that is likely to be met in an engineering problem is Fourier expandable, and its Fourier series will yield the value of the function at every point of continuity and yield the average value at points of discontinuity (see Section 26).

PROBLEMS

1. Establish the second and third orthogonality conditions of Equations (68).

2. Expand the function $f(x)$,

$$\begin{aligned} f(x) &= 4 & -\pi < x < 0, \\ f(x) &= 1 & 0 < x < \pi, \end{aligned}$$

given in the full Fourier interval $(-\pi, \pi)$ into a Fourier series.

3. Obtain the Fourier series expansion for the function defined in $(-\pi, \pi)$ by Figure 37.

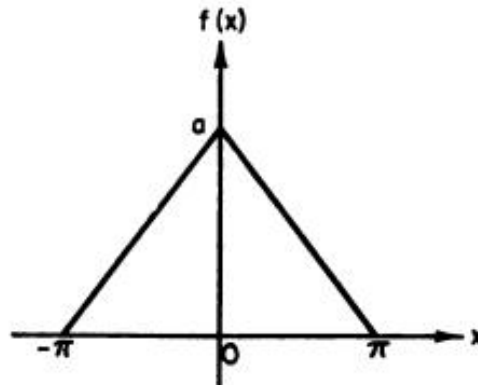


Fig. 37

4. Write the Fourier series expansion for the function $f(x)$,

$$f(x) = x^2, \quad -\pi < x < \pi.$$

5. Expand the function $f(x)$,

$$\begin{aligned} f(x) &= 0 & -\pi < x \leq 0, \\ f(x) &= x & 0 \leq x < \pi, \end{aligned}$$

in a Fourier series.

6. Find the value of the Fourier series of Problems 2, 3, 4, and 5 at the points $x = -\pi, 0, \pi, 2\pi$.

7. Determine whether the following functions (considered as real functions of the real variable x) are odd, even, or neither.

- (a) $\sec x$.
- (b) x^3 .
- (c) $\sinh x$.
- (d) $\tan x^2$.
- (e) $f(x^2)$, f an arbitrary function.
- (f) $\phi(\sin x)$, ϕ an arbitrary function.

8. Prolong the function $f(x)$ given in Figure 38 into the half Fourier interval so that it becomes an odd function, and obtain its Fourier series expansion.

9. Extend the function of Figure 38 into the interval $(-\pi, 0)$ such that its Fourier series expansion contains only cosine terms. Obtain this expansion.

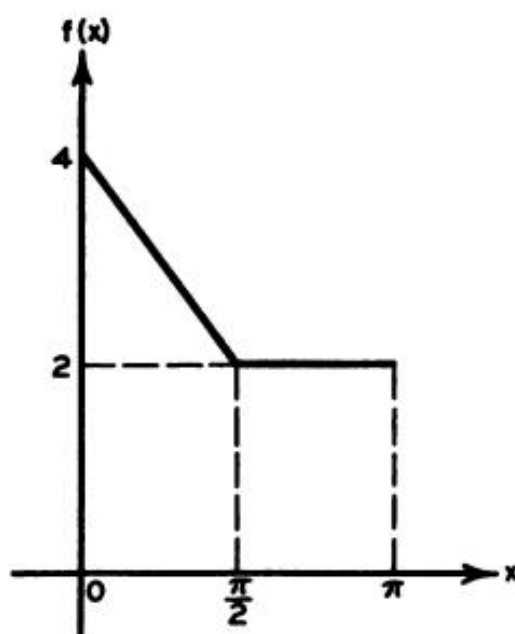


Fig. 38

10. Extend the function

$$f(x) = 2 \sin 3x \quad 0 < x < \pi,$$

into the half Fourier interval $(-\pi, 0)$ such that it becomes an odd function, and obtain the corresponding Fourier series expansion.

11. Expand the function $f(x)$,

$$f(x) = \cos^2 x \quad -\pi < x < \pi,$$

into a Fourier series.

12. We are interested in the values of the function

$$f(x) = x^3 + 2x - 4$$

in the interval $(-\pi, \pi)$. Obtain a Fourier series expansion that will represent $f(x)$ in this interval.

13. Let $f(x)$ be a periodic function of period 2π defined in the full Fourier interval by the equation

$$f(x) = x \quad -\pi < x < \pi.$$

Calculate the best trigonometric approximation to $f(x)$ using

- (a) One sine term.
- (b) Two sine terms.
- (c) Three sine terms.
- (d) Four sine terms.

Plot the results of (a), (b), (c), and (d) in the interval $(0, \pi)$.

14. A function $f(x)$ is defined in the *full* Fourier interval $(0, 2c)$ by the equations

$$y = \frac{h}{c}x \quad 0 < x < c,$$

$$y = h - \frac{h}{c}(x - c) \quad c < x < 2c.$$

Obtain its Fourier series expansion.

15. Expand the function $f(x)$ given in Figure 39 below into a Fourier series.

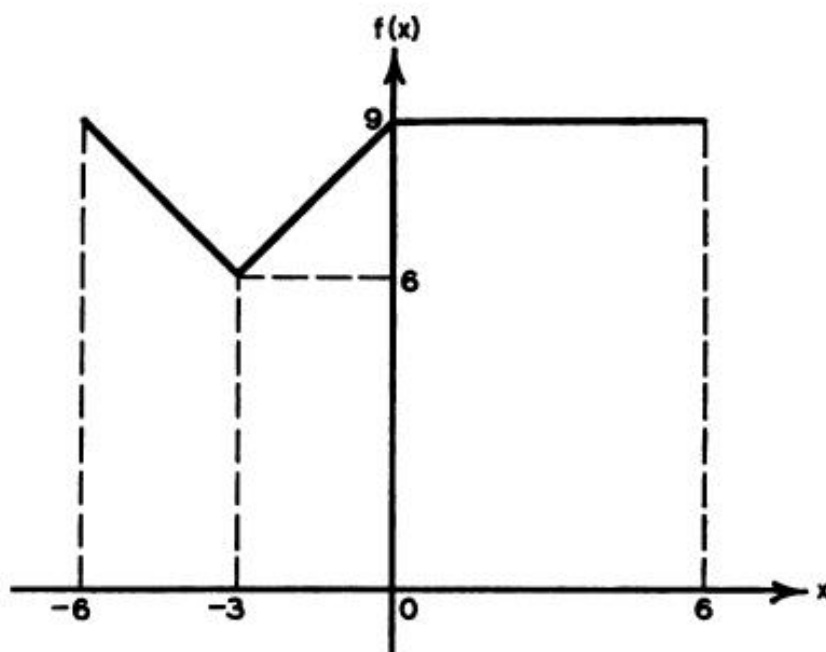


Fig. 39

16. A voltage $e(t) = E_0 \sin \omega t$ is passed through a half-wave rectifier which clips the negative portions of the voltage. Obtain a Fourier series expansion for the output voltage of the rectifier.

17. If the voltage $e(t) = E_0 \sin \omega t$ is passed through a full-wave rectifier, obtain its Fourier series expansion.

18. Establish the formulas of Section 31 of the text.

19. Expand the function given in Figure 34, Section 33 of the text as an even function.

20. Extend the function $f(x)$ given in Figure 40 so that it becomes an odd function in the full Fourier interval $(-2, 2)$.

21. Prolong the function of Figure 40 so that its Fourier series expansion contains only cosine terms.

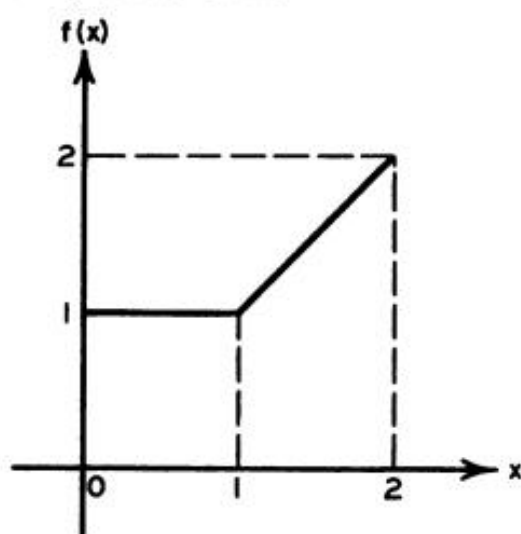


Fig. 40

22. Evaluate the Fourier series of Problems 20 and 21 at the points $x = 0, 1, 2, 3$.

23. Write the Fourier series corresponding to the function

$$f(x) = \frac{h}{c}x \quad 0 < x < c,$$

$$f(x) = h - \frac{h}{c}(x - c) \quad c < x < 2c,$$

using sines only.

24. Write the Fourier series corresponding to the function of Problem 23, using cosines only.

25. Obtain the Fourier series expansion for the periodic function shown in Figure 41.

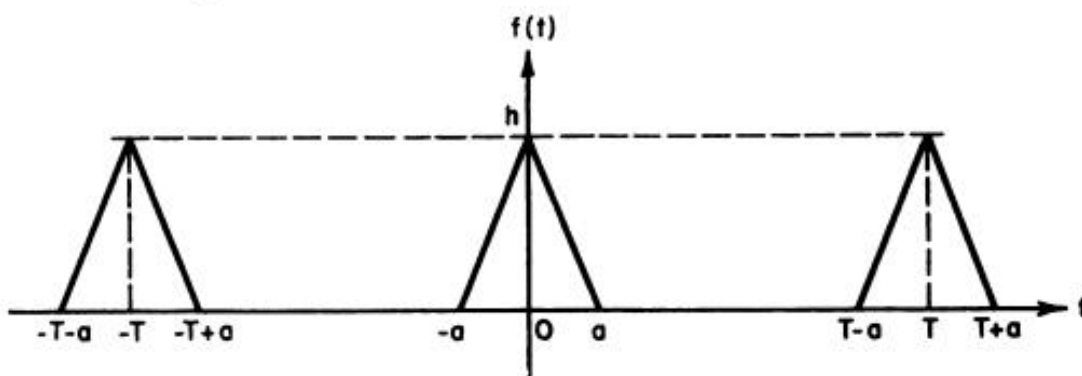


Fig. 41

26. Expand the function

$$\begin{aligned} f(x) &= 0 & -2 < x < 0, \\ f(x) &= 3x & 0 < x < 2, \end{aligned}$$

into a sine Fourier series involving a phase angle.

27. Expand the function

$$\begin{aligned} f(x) &= -x & -3 < x < 0, \\ f(x) &= 0 & 0 < x < 3, \end{aligned}$$

into a cosine Fourier series involving a phase angle.

28. Expand the function $f(x) = e^x$ given in the interval $0 < x < 2$ into a complex Fourier series.

29. Extend the function $f(x)$ of Problem 28 such that it is periodic of period 4 and an even function, and obtain its complex Fourier series expansion.

30. Write the exponential Fourier series for the function $f(x)$ of Problem 28 after prolonging it in such a way that it is periodic of period 4 and an odd function.

31. Expand the function given in Figure 42 below in a complex Fourier series.

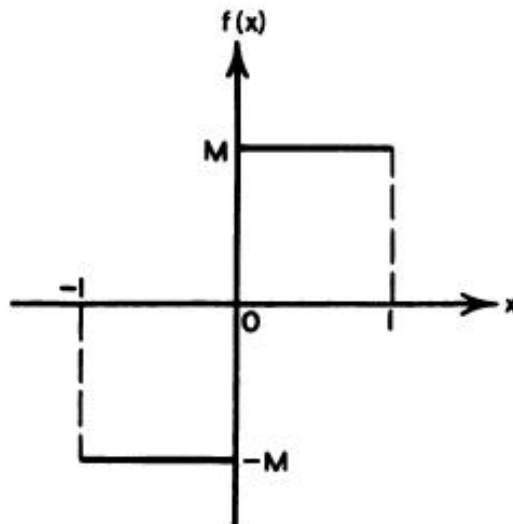


Fig. 42

32. Write the complex Fourier series expansion for the function

$$\begin{aligned} f(x) &= -x & -1 < x \leq 0, \\ f(x) &= x^2 & 0 \leq x < 1. \end{aligned}$$

33. Write the double Fourier series for a function $f(x,y)$ of two variables in such a manner that the same formula can be used for a_{00} , a_{0m} , a_{n0} , and a_{nm} ; the same formula for b_{0m} and b_{nm} ; the same formula for c_{n0} and c_{nm} .

34. Write the double Fourier series for the function $f(x,y)$ defined analytically in the full Fourier interval by the equation

$$\begin{aligned} f(x,y) &= xy & -\pi < x < \pi, \\ & & -\pi < y < \pi. \end{aligned}$$

35. Let $f(x,y)$ be defined as

$$\begin{aligned} f(x,y) &= x^2y^2 & 0 < x < \pi, \\ & & 0 < y < \pi, \end{aligned}$$

in the quarter Fourier rectangle. Extend it as an odd function to the full Fourier rectangle and obtain its Fourier series expansion.

36. Obtain the Fourier series expansion (analogous to Equations 120-127 of the text) if :

- (a) $f(x,y)$ is an even function in both x and y .
- (b) $f(x,y)$ is an even function of x and an odd function of y .
- (c) $f(x,y)$ is an odd function of x and an even function of y .
- (d) $f(x,y)$ is an odd function of x .
- (e) $f(x,y)$ is an even function of x .
- (f) $f(x,y)$ is an odd function of y .
- (g) $f(x,y)$ is an even function of y .

37. Extend the function of Problem 35 so that it becomes an even function in both x and y , and obtain its double Fourier series expansion.

38. Extend the function of Problem 35 so that it becomes an even function of x and an odd function of y , and obtain its Fourier series expansion.

39. Extend the function of Problem 35 so that it becomes an odd function of x and an even function of y , and obtain its Fourier series expansion.

40. Expand the function in Figure 43 into a double Fourier series.

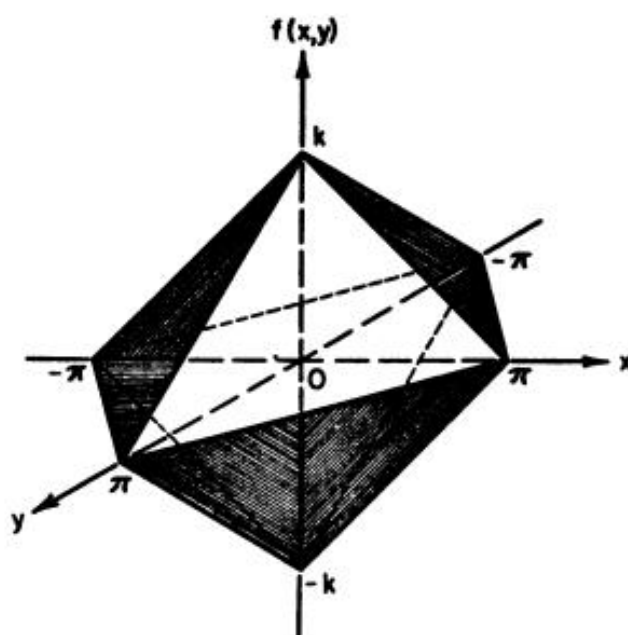


Fig. 43

41. Given the function $f(x, y)$ defined in the quarter Fourier rectangle as

$$f(x, y) = x^2 y \quad \begin{array}{l} 0 < x < \pi, \\ 0 < y < \pi. \end{array}$$

Extend this function to the full Fourier rectangle $(-\pi < x < \pi, -\pi < y < \pi)$ such that it becomes an odd function of x and y , and obtain its Fourier series expansion.

42. Extend the function of Problem 41 so that it becomes a function

- (a) Even in x and odd in y ,
- (b) Even in y and odd in x ,
- (c) Even in x and even in y ,

and obtain its Fourier series expansion.

43. Generalize the double Fourier series expansion from the square $(-\pi < x < \pi, -\pi < y < \pi)$ to an arbitrary rectangle $(-a < x < a, -b < y < b)$.

44. Obtain the analog of the complex Fourier series for functions of two variables.

45. Deduce the form of the Fourier series and the equations for the coefficients of a function $f(x, y, z)$ which is periodic of period 2π in x , y , and z , and an odd function in all three variables.

46. Determine whether the following functions can be expanded into a Fourier series or not and give the reasons why.

(a) $\sin \frac{1}{x}$ defined in $-\pi < x < \pi$.

(b) $\sin \frac{1}{x}$ defined in $1 < x < 2$.

(c) $\frac{1}{1+x}$ defined in $-2 < x < 2$.

(d) $\log x$ defined in $1 < x < 4$.

(e) $f(x) = 1 - 2^{-(n+1)}$ for $1 - 2^{-n} < x < 1 - 2^{-(n+1)}$, $n = 0, 1, 2, \dots$,
where $f(x)$ is defined in $0 < x < 1$.

Chapter III

SEPARATION OF VARIABLES

41. Introduction

In Chapter I we derived various partial differential equations of interest in engineering and physics. In Chapter II the elementary theory of Fourier series was developed. In the present chapter we shall combine these results; that is, we shall utilize Fourier series to solve the differential equations derived in Chapter I. This will be accomplished by the powerful but elementary method of "separation of variables."

It is not immediately evident, but it will be shown later (Chapter VI) that linear partial differential equations of the second order fall into essentially three distinct types, of which the following are typical examples :

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2} = 0, \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$
$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{K} \frac{\partial u}{\partial t} = 0.$$

The first might represent the equation of the vibrating string, in which case $a^2 = Tg/w$. The second might represent the temperature distribution in a plate, in which the temperature is independent of time. The third might represent the voltage in a transmission line of negligible inductance and conductance, in which case $1/K = RC$ is the time constant. We shall show that all three of the above equations with suitable boundary conditions, can be solved by the method of separation of variables.

Since most of the equations derived in Chapter I were of the second order in two independent variables we have stressed this type in the present chapter. However, separation of variables

also applies to equations whose order is higher than the second (Section 51) and to equations in more than two independent variables (Section 52).

42. Solution of the vibrating string problem

As a first example we shall consider the problem of the vibrating string (see Figure 44). This figure represents a tightly stretched

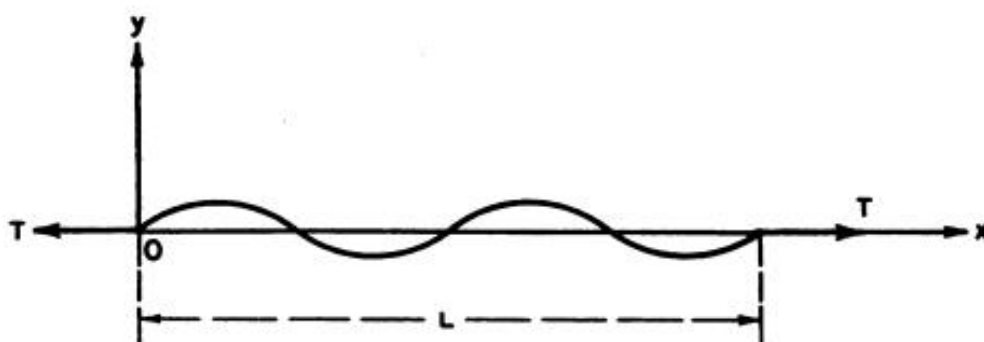


Fig. 44

string fixed at the end points 0, L , and subjected to a uniform tension of T pounds. We have seen earlier that the motion of the string is governed by the equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2}, \quad (129)$$

where $a^2 = Tg/w$, with w the weight per unit length of the string, and g the gravitational constant. Typical boundary conditions were also determined to be

1. $u(0, t) = 0$,
 2. $u(L, t) = 0$,
 3. $\frac{\partial}{\partial t} u(x, 0) = 0$,
 4. $u(x, 0) = f(x) \neq 0$,
- (130)

where we have assumed the initial position of the string to be $f(x)$. [If $f(x) = 0$ the only solution to the boundary value problem is $u(x, t) = 0$.]

The main assumption in separation of variables consists in

stating that the solution of Equation (129) can be written in the form of a function of x times a function of t ,

$$u(x,t) = X(x) T(t). \quad (131)$$

This assumption is justified by the fact that it yields solutions to a variety of engineering problems.

If we substitute Equation (131) into Equation (129), we obtain

$$XT'' = a^2 X''T \quad \text{or} \quad \frac{X''}{X} = \frac{T''}{a^2 T}$$

where the primes on the functions X and T represent differentiation with respect to the only variable present. Now the right-hand side of the above equation is independent of x and the left-hand side is independent of t . Since they are equal, their common value cannot be a function of t or x , and must therefore be a constant, say λ :

$$\frac{X''}{X} = \lambda, \quad \frac{T''}{a^2 T} = \lambda.$$

Hence

$$X'' - \lambda X = 0, \quad T'' - \lambda a^2 T = 0.$$

The partial differential equation of Equation (129) has thus been reduced to two *ordinary* differential equations.

It will be convenient to determine whether λ is a positive negative, or zero constant before we solve the above equations. (See Problem 20 at the end of this chapter.) If $\lambda > 0$, the solutions of $X'' - \lambda X = 0$ will be

$$e^{-\sqrt{\lambda}x} \quad \text{and} \quad e^{+\sqrt{\lambda}x},$$

that is, exponentials. This assumption contradicts the physical fact that the motion of the string is oscillatory in the y -direction. The assumption that $\lambda = 0$ leads to the solutions,

$$X(x) = Ax + B, \quad T(t) = Ct + D,$$

from which by Equation (131),

$$u(x,t) = (Ax + B)(Ct + D),$$

where A , B , C , D are arbitrary constants. Such a solution is clearly not oscillatory and furthermore could not be made to satisfy the boundary conditions. Our remaining alternative is

to assume $\lambda = -\beta^2 < 0$, where β is a real number. In this case the general solution of

$$X'' + \beta^2 X = 0$$

is

$$X(x) = A \cos \beta x + B \sin \beta x.$$

To determine the arbitrary constant A and B , notice that, since from the first boundary condition

$$0 = u(0, t) = X(0) T(t)$$

and $T(t)$ is not identically zero, $X(0) = 0$. Similarly the second condition

$$0 = u(L, t) = X(L) T(t)$$

implies $X(L) = 0$. From $X(0) = 0$ we conclude that

$$X(0) = A \cos 0 + B \sin 0 = A = 0,$$

and hence that

$$X(x) = B \sin \beta x.$$

From the second condition

$$X(L) = B \sin \beta L = 0.$$

But if $B = 0$, $X(x) \equiv 0$ and hence $u(x, t) \equiv 0$, since $u(x, t) = X(x) T(t)$. This contradicts the assumption that $u(x, 0) = f(x) \neq 0$. We conclude therefore that

$$\sin \beta L = 0,$$

from which

$$\beta L = n\pi, \quad \beta = \frac{n\pi}{L}, \quad n = 1, 2, 3, \dots$$

Hence for every integral value of n we get a value of β and a different solution,

$$X_n(x) = B_n \sin \frac{n\pi x}{L}.*$$

* There is no necessity for letting n be negative. For suppose we had a term

$$X_{-n}(x) = C_{-n} \sin \frac{(-n)\pi x}{L}$$

in addition to

$$X_n(x) = B_n \sin \frac{n\pi x}{L}.$$

Then we could write their sum, say $\bar{X}_n$, where n is a positive integer as

$$\bar{X}_n(x) = X_n(x) + X_{-n}(x) = (B_n - C_{-n}) \sin \frac{n\pi x}{L} = \bar{B}_n \sin \frac{n\pi x}{L}$$

—which is identical in form with $X_n(x)$.

Consider now the equation on $T(t)$,

$$T'' + \beta^2 a^2 T = 0.$$

Its general solution is

$$T(t) = C \cos \beta at + D \sin \beta at,$$

and its derivative is

$$T'(t) = -\beta a C \sin \beta at + \beta a D \cos \beta at.$$

From the third boundary condition

$$\frac{\partial}{\partial t} u(x, 0) = 0$$

we conclude that

$$X(x) T'(0) = 0,$$

and hence that $T'(0) = 0$. From this fact

$$T'(0) = -\beta a C \sin 0 + \beta a D \cos 0 = \beta a D = 0,$$

or $D = 0$. Therefore

$$T(t) = C \cos \beta at.$$

Since $\beta = n\pi/L$, we have a solution $T_n(t)$ for each integral value of n :

$$T_n(t) = C_n \cos \frac{n\pi a}{L} t,$$

and for every integral value of n , $n = 1, 2, 3, \dots$,

$$X_n(x) T_n(t) = b_n \sin \frac{n\pi}{L} x \cos \frac{n\pi a}{L} t \quad (b_n = B_n C_n)$$

is a solution of the partial differential equation of Equation (129), which satisfies the first three boundary conditions of Equation (130) whatever the value of the constant b_n . Since Equation (129) is *linear*, any linear combination of solutions is a solution and hence the series

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x \cos \frac{n\pi a}{L} t$$

is a function which formally satisfies the differential equation and three of the boundary conditions. There remains one more boundary condition to be satisfied,

$$u(x, 0) = f(x).$$

To satisfy this condition the b_n must be determined such that

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L}x. \quad (132)$$

It is thus seen that the b_n are the sine Fourier coefficients of $f(x)$, and hence are given by

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L}x \, dx.$$

The solution of the problem of the vibrating string with the given initial conditions is therefore

$$u(x,t) = \frac{2}{L} \sum_{n=1}^{\infty} \left[\int_0^L f(x) \sin \frac{n\pi}{L}x \, dx \right] \sin \frac{n\pi}{L}x \cos \frac{n\pi a}{L}t. \quad (133)$$

43. Alternate form for the solution of the vibrating string

A suitable trigonometric identity transforms Equation (133) into a very elegant form. Remembering that

$$\sin A \cos B = \frac{1}{2} [\sin (A - B) + \sin (A + B)],$$

we may write

$$\sin \frac{n\pi x}{L} \cos \frac{n\pi a}{L}t = \frac{1}{2} \left[\sin \frac{n\pi}{L}(x - at) + \sin \frac{n\pi}{L}(x + at) \right],$$

and Equation (133) becomes

$$u(x,t) = \frac{1}{2} \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L}(x - at) + \frac{1}{2} \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L}(x + at). \quad (134)$$

But

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L}x,$$

and hence if we let $x = x - at$ in the first sum of Equation (134), and $x = x + at$ in the second sum, we may write

$$u(x,t) = \frac{f(x - at) + f(x + at)}{2}. \quad (135)$$

In this formula we must remember that $f(x \pm at)$ represents the periodic prolongation of $f(x)$. Since at has the dimensions of a displacement, the function $f(x - at)$ represents the function $f(x)$ displaced at units to the right (see Figure 45). Similarly $f(x + at)$

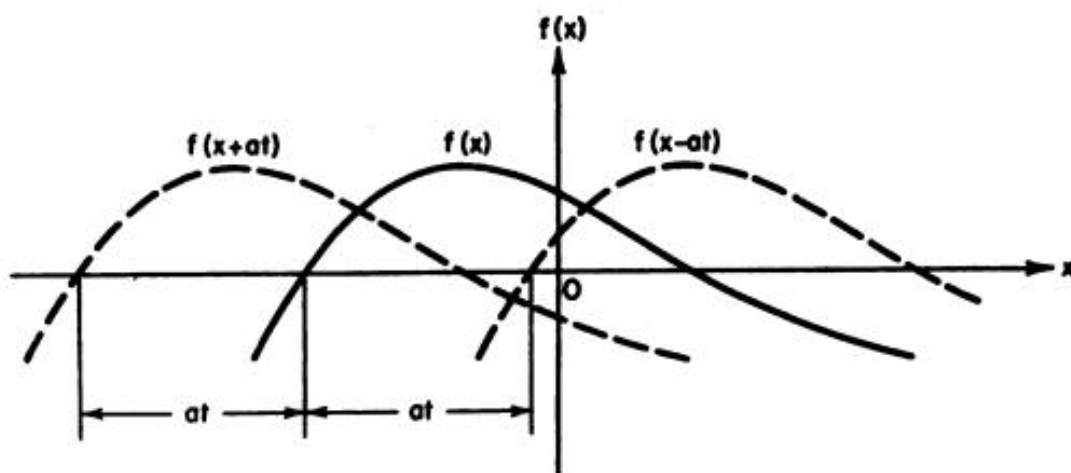


Fig. 45

represents the function $f(x)$ translated at units to the left, and we can therefore think of $f(x \pm at)$ as representing a traveling wave of shape $f(x)$ with velocity $\mp a$. This is the basis for calling the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0$$

(and its generalizations), the *wave equation*.

44. An example of the wave equation

We shall consider a numerical example of integration of the wave equation. Assume that the initial displacement of string is given by

$$f(x) = \frac{x(10 - x)}{1000}$$

where $L = 10$ and $a^2 = 10,000$. [We note that $f(0) = f(10) = 0$.] The Fourier series solution of this problem is given by Equation (133), where the Fourier coefficients b_n are given by

$$b_n = \frac{1}{5000} \int_0^{10} x(10 - x) \sin \frac{n\pi}{10} x \, dx.$$

We leave it to the reader to verify that

$$b_n = 0 \quad \text{if } n \text{ is even,}$$

$$b_n = \frac{4}{5\pi^2 n^2} \quad \text{if } n \text{ is odd,}$$

and hence that

$$u(x,t) = \frac{4}{5\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^2} \sin \frac{n\pi}{10} x \cos 10n\pi t.$$

From what we have seen in the previous section we may write $u(x,t)$ in the closed form

$$u(x,t) = \frac{f(x-100t) + f(x+100t)}{2}.$$

The values of $1000 u(x,t)$ obtained by taking values of x at the points $x = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10 = L$ and letting $t = 0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.06$ seconds appear in the table below.

$x \backslash t$	0	0.01	0.02	0.03	0.04	0.05	0.06
0	0	0	0	0	0	0	0
1	9	8	6	4	2	0	-2
2	16	15	12	8	4	0	-4
3	21	20	17	12	6	0	-6
4	24	23	20	15	8	0	-8
5	25	24	21	16	9	0	-9
6	24	23	20	15	8	0	-8
7	21	20	17	12	6	0	-6
8	16	15	12	8	4	0	-4
9	9	8	6	4	2	0	-2
10	0	0	0	0	0	0	0

In computing these values one must remember that $f(x)$ has been expanded into a sine series and hence $f(-x) = -f(x)$, and that with $a = 100$ and

$$t = \frac{m}{100} \quad (m = 0, 1, 2, \dots)$$

the tabular values are equal to

$$\frac{1}{2}[f(x+m) + f(x-m)].$$

The results are plotted in Figure 46. Note that the period of the vibration is 0.2 second, that is, the value of t for which

$$\frac{\pi a}{L}t = 10\pi t = 2\pi.$$

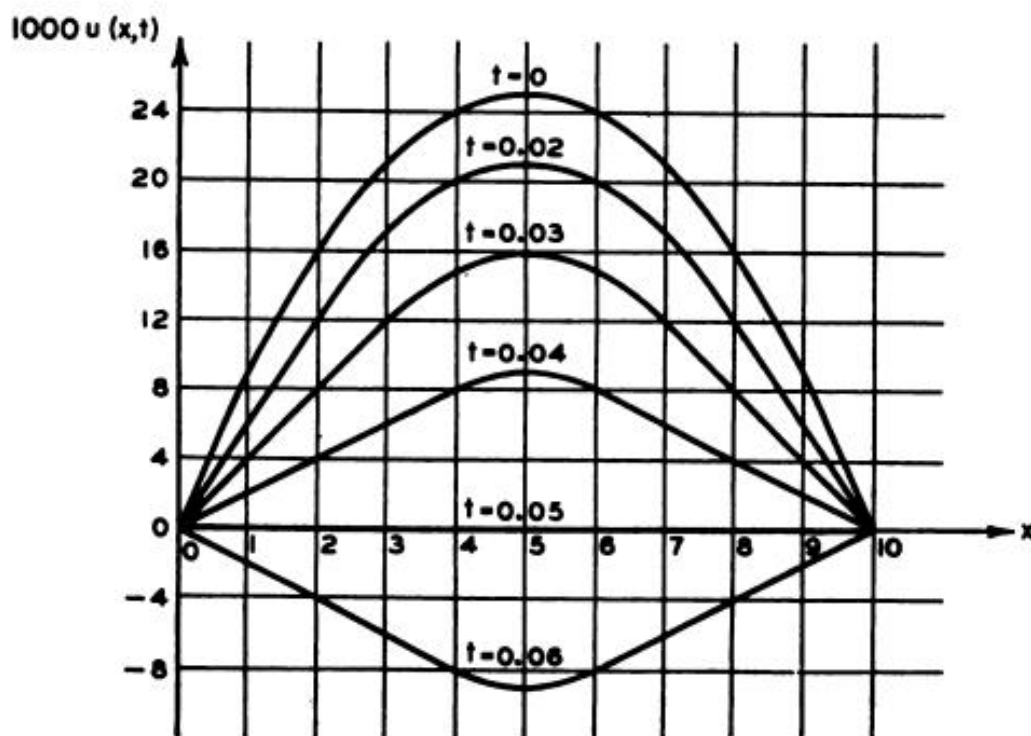


Fig. 46

45. Temperature distribution in a slab

As a second example of the application of the method of separation of variables we shall consider the two-dimensional Laplacian $\nabla^2 u = 0$. As was mentioned in Section 41, the equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (136)$$

might represent, for example, the steady-state temperature distribution in a plate insulated on its faces. Consider a rectangular

plate of sides a and b (Figure 47) on three sides of which the temperature is assumed to be zero, while the temperature on the remaining side is a specified function of x , namely $f(x)$. We are

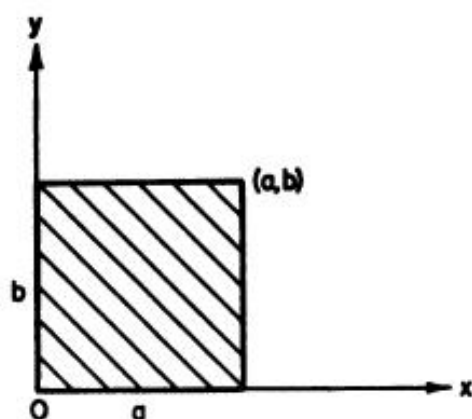


Fig. 47

interested in the temperature $u(x,y)$ in the plate which satisfies Equation (136) and the boundary conditions

$$\begin{aligned} 1. \quad u(0,y) &= 0, \\ 2. \quad u(a,y) &= 0, \\ 3. \quad u(x,b) &= 0, \\ 4. \quad u(x,0) &= f(x). \end{aligned} \tag{137}$$

As before, assume a solution of the form

$$u(x,y) = X(x) Y(y)$$

and substitute it in Equation (136), obtaining

$$X''Y + XY'' = 0$$

or

$$\frac{X''}{X} = -\frac{Y''}{Y}.$$

The left-hand side of this equation is independent of y , and the right-hand side is independent of x ; therefore their value is a constant, say λ . Thus

$$X'' - \lambda X = 0, \quad Y'' + \lambda Y = 0.$$

We shall assume $\lambda = -\beta^2 < 0$ and show that a negative value of λ allows the satisfaction of all the boundary conditions. With $\lambda = -\beta^2$ the general solution of

$$X'' + \beta^2 X = 0$$

is

$$X(x) = A \cos \beta x + B \sin \beta x.$$

From the first boundary condition

$$0 = u(0,y) = X(0) Y(y),$$

and since $Y(y) \neq 0$, we find that

$$X(0) = 0, \quad \text{or} \quad X(0) = A = 0.$$

Similarly the second boundary condition

$$0 = u(a, y) = X(a) Y(y)$$

implies $X(a) = 0$, or

$$X(a) = B \sin \beta a = 0.$$

Since $B = 0$ would imply $u(x, y) \equiv 0$, we must have $\sin \beta a = 0$, or

$$\beta a = n\pi, \quad \beta = \frac{n\pi}{a}, \quad n = 1, 2, \dots$$

With these values of β ,

$$X_n(x) = B_n \sin \frac{n\pi}{a} x.$$

The solution of the equation for $Y(y)$ yields

$$Y(y) = C \cosh \beta y + D \sinh \beta y$$

or, recalling that β is a function of n ,

$$Y_n(y) = C_n \cosh \frac{n\pi}{a} y + D_n \sinh \frac{n\pi}{a} y.$$

It is convenient to write $Y_n(y)$ in the equivalent form

$$Y_n(y) = E_n \sinh \frac{n\pi}{a} (y - y_0)$$

and to notice that, since from the third boundary condition

$$0 = u(x, b) = X(x) Y(b)$$

and $X(x) \neq 0$, $Y(b)$ must vanish :

$$Y_n(b) = E_n \sinh \frac{n\pi}{a} (b - y_0) = 0.$$

Hence

$$y_0 = b,$$

and the functions

$$u_n(x, y) = b_n \sin \frac{n\pi}{a} x \sinh \frac{n\pi}{a} (y - b)$$

satisfy the partial differential equation of Equation (136) and the first three boundary conditions of Equation (137), whatever the constants b_n . Since Equation (136) is linear, any linear combination of the u_n , such as

$$u(x,y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{a}x \sinh \frac{n\pi}{a}(y-b)$$

is also a formal solution of Equation (136) satisfying three of the boundary conditions. In order that $u(x,y)$ satisfy the fourth boundary condition, we must choose the b_n such that

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sinh \left(-\frac{n\pi b}{a} \right) \sin \frac{n\pi}{a}x$$

and hence $-b_n \sinh (n\pi b/a)$ are the sine Fourier coefficients of $f(x)$:

$$-b_n \sinh \frac{n\pi b}{a} = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi}{a}x dx.$$

The b_n are therefore given by

$$b_n = \frac{-2}{a \sinh (n\pi b/a)} \int_0^a f(x) \sin \frac{n\pi}{a}x dx,$$

and the final solution of the problem reads

$$u(x,y) = -\frac{2}{a} \sum_{n=1}^{\infty} \left[\frac{1}{\sinh (n\pi b/a)} \int_0^a f(x) \sin \frac{n\pi x}{a} dx \right] \sin \frac{n\pi}{a}x \sinh \frac{n\pi}{a}(y-b). \quad (138)$$

46. Example of the heat equation

As an application of the solution of the previous section, assume

$$f(x) = T \quad (\text{a constant}).$$

Then

$$b_n = \frac{-2T}{a \sinh (n\pi b/a)} \int_0^a \sin \frac{n\pi}{a}x dx$$

and

$$\begin{aligned} b_n &= 0 && \text{if } n \text{ is even,} \\ b_n &= \frac{-4T}{n\pi \sinh (n\pi b/a)} && \text{if } n \text{ is odd.} \end{aligned}$$

Hence by Equation (138),

$$u(x,y) = -\frac{4T}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{\sin(n\pi/a)x \sinh[(n\pi/a)(y-b)]}{n \sinh(n\pi b/a)}.$$

If the plate is square ($a = b$) and we wish, for example, to find the temperature at the center, $x = a/2$, $y = a/2$, we obtain

$$u\left(\frac{a}{2}, \frac{a}{2}\right) = \frac{2T}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{(-1)^{(n-1)/2}}{n \cosh(n\pi/2)}.$$

It can be shown that the sum of this series is $\pi/8$ and hence that

$$u\left(\frac{a}{2}, \frac{a}{2}\right) = \frac{T}{4}.$$

47. Voltage in a transmission line

The voltage $e(x,t)$ in a transmission line (of the submarine cable type), grounded at $x = 0$ and $x = L$, and with an initial voltage distribution $f(x)$, was shown in Section 9 to satisfy the equation

$$\frac{\partial^2 e}{\partial x^2} = \frac{1}{K} \frac{\partial e}{\partial t}, \quad \frac{1}{K} = RC, \quad (139)$$

and the boundary conditions (see Figure 48)

$$\begin{aligned} 1. \quad & e(0,t) = 0, \\ 2. \quad & e(L,t) = 0, \\ 3. \quad & e(x,0) = f(x). \end{aligned} \quad (140)$$

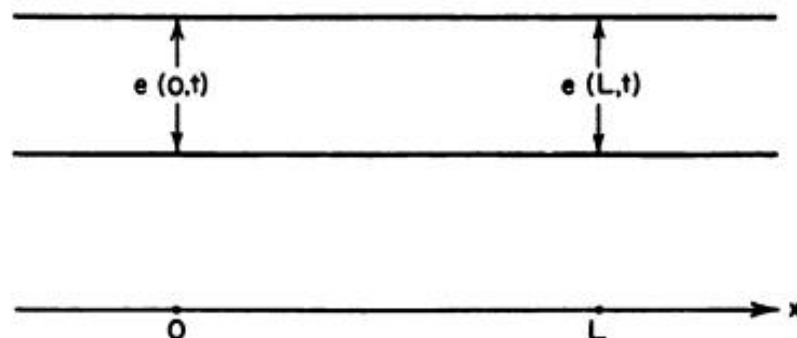


Fig. 48

If we assume a solution of the form

$$e(x,t) = X(x) T(t),$$

Equation (139) yields

$$KX''T = XT' \quad \text{or} \quad \frac{X''}{X} = \frac{T'}{KT} = \lambda.$$

Hence

$$X'' - \lambda X = 0, \quad T' - K\lambda T = 0.$$

If $\lambda > 0$, the solutions of $X'' - \lambda X = 0$ are exponential in character, which is physically unreasonable. Hence we assume

$$\lambda = -\beta^2,$$

and the equations above become

$$X'' + \beta^2 X = 0, \quad T' + \beta^2 KT = 0.$$

The solution to the first of these equations is

$$X(x) = A \cos \beta x + B \sin \beta x.$$

From the first boundary condition,

$$\phi(0, t) = X(0) T(t) = 0,$$

we conclude that

$$X(0) = 0,$$

and hence

$$X(0) = A \cos 0 + B \sin 0 = A = 0.$$

From the second boundary condition,

$$\phi(L, t) = X(L) T(t) = 0,$$

we obtain

$$X(L) = 0$$

and hence

$$X(L) = B \sin \beta L = 0,$$

from which, as before,

$$\beta L = n\pi, \quad \beta = \frac{n\pi}{L}, \quad n = 1, 2, 3, \dots$$

and

$$X_n(x) = B_n \sin \frac{n\pi x}{L}.$$

The solution of

$$T' + \beta^2 KT = 0 \quad \text{is} \quad T = Ce^{-\beta^2 Kt}$$

or

$$T_n(t) = C_n e^{-(n\pi/L)^2 K t},$$

by means of which, due to the linearity of Equation (139) we obtain the function

$$e(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) e^{-(n\pi/L)^2 K t}$$

satisfying the partial differential equation and the first two boundary conditions. This function satisfies the third boundary condition if

$$e(x,0) = f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L}x.$$

Hence the b_n are the sine Fourier coefficients of $f(x)$:

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L}x \, dx.$$

The complete solution of the transmission line problem thus becomes

$$e(x,t) = \frac{2}{L} \sum_{n=1}^{\infty} \left[\int_0^L f(x) \sin \frac{n\pi x}{L} dx \right] \sin\left(\frac{n\pi}{L}x\right) e^{-(n\pi/L)^2 K t}.$$

48. Example of the transmission line equation

If we assume

$$f(x) = E \quad (\text{a constant}),$$

then

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L E \sin \frac{n\pi}{L}x \, dx \\ &= 0 \quad \text{if } n \text{ is even} \\ &= \frac{4E}{n\pi} \quad \text{if } n \text{ is odd,} \end{aligned}$$

and

$$e(x,t) = \frac{4E}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{L}x\right) e^{-(n\pi/L)^2 K t}.$$

The voltage approaches zero as $t \rightarrow \infty$ for any x due to the negative exponential. Also, for fixed t , the voltage is sinusoidal in nature. Both these phenomena agree with our physical intuition. While we are interested only in $e(x,t)$ for x between 0 and L , since that is the length of the transmission line, the solution is valid for any value of x . The value of $e(x,t)$ at the point x_0 plus or minus integral multiples of $2L$ is the same as $e(x_0,t)$. This is due, of course, to the fact that $f(x) = E$ is actually a periodic function which coincides with E in the fundamental interval $(0,L)$.

49. Nonhomogeneous boundary conditions

In each of the three previous examples, only *one* of the boundary conditions was different from zero, but this is not an essential restriction on the method of separation of variables, as will be shown by means of the following example.

Consider the equation of the vibrating string,

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2} = 0, \quad (141)$$

when the string is fixed at the end points but has an initial velocity $g(x)$ as well as an initial displacement $f(x)$. The boundary conditions become in this case

$$\begin{aligned} 1. \quad & u(0,t) = 0 \\ 2. \quad & u(L,t) = 0 \\ 3. \quad & \frac{\partial}{\partial t} u(x,0) = g(x) \\ 4. \quad & u(x,0) = f(x). \end{aligned} \quad (142)$$

We assume as in Section 42 a solution of the form

$$u(x,t) = X(x) T(t)$$

and obtain the two ordinary differential equations

$$X'' + \beta^2 X = 0, \quad T'' + \beta^2 a^2 T = 0.$$

The solution of the first of these equations is

$$X(x) = A \cos \beta x + B \sin \beta x,$$

and from the first two boundary conditions of Equations (142) we conclude, as before, that $A = 0$ and

$$\beta = \frac{n\pi}{L}, \quad n = 1, 2, \dots$$

Since the third boundary condition does not eliminate one of the arbitrary constants appearing in the solution of $T'' + \beta^2 a^2 T = 0$, we have now the solutions

$$u_n(x, t) = b_n \sin \frac{n\pi}{L} x \cos \frac{n\pi a}{L} t + c_n \sin \frac{n\pi}{L} x \sin \frac{n\pi a}{L} t,$$

which satisfy the partial differential equation of Equation (141) and the first two boundary conditions of Equations (142) whatever the values of b_n and c_n . Hence the linear combination

$$u(x, t) = \sum_{n=1}^{\infty} \left(b_n \sin \frac{n\pi}{L} x \cos \frac{n\pi a}{L} t + c_n \sin \frac{n\pi}{L} x \sin \frac{n\pi a}{L} t \right) \quad (143)$$

also has this property. Computing the time derivative of this function,

$$\frac{\partial}{\partial t} u(x, t) = \sum_{n=1}^{\infty} \left(-b_n \frac{n\pi a}{L} \sin \frac{n\pi}{L} x \sin \frac{n\pi a}{L} t + c_n \frac{n\pi a}{L} \sin \frac{n\pi}{L} x \cos \frac{n\pi a}{L} t \right),$$

and substituting it in the third boundary condition we obtain

$$\frac{\partial}{\partial t} u(x, 0) = g(x) = \sum_{n=1}^{\infty} \frac{n\pi a}{L} c_n \sin \frac{n\pi}{L} x.$$

This equation shows that the constants $(n\pi a/L)c_n$ are the sine Fourier coefficients of $g(x)$, so that

$$c_n = \frac{2}{n\pi a} \int_0^L g(x) \sin \frac{n\pi}{L} x \, dx.$$

With these values for the c_n , $u(x, t)$ as given by Equation (143) will satisfy the partial differential equation of Equation (141) as well as the first three boundary conditions of Equations (142). Similarly, substituting Equation (143) into the last condition,

$$u(x, 0) = f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x,$$

and we see that the b_n are the sine Fourier coefficients of $f(x)$:

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx.$$

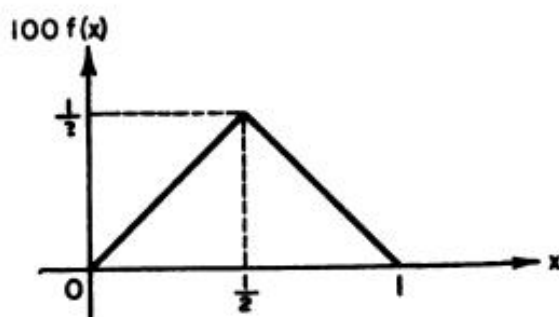
Hence the function

$$\begin{aligned} u(x,t) = \sum_{n=1}^{\infty} \left\{ \left[\frac{2}{L} \int_0^L f(x) \sin \frac{n\pi}{L} x dx \right] \sin \frac{n\pi}{L} x \cos \frac{n\pi a}{L} t \right. \\ \left. + \left[\frac{2}{n\pi a} \int_0^L g(x) \sin \frac{n\pi}{L} x dx \right] \sin \frac{n\pi}{L} x \sin \frac{n\pi a}{L} t \right\} \quad (144) \end{aligned}$$

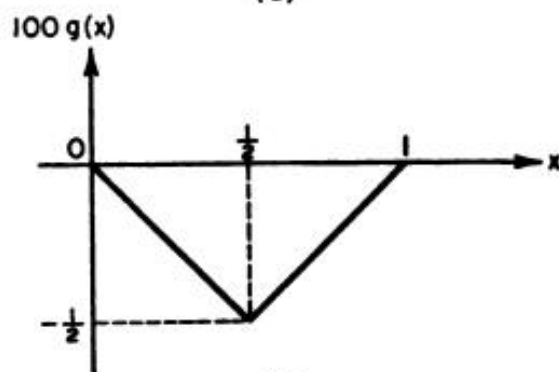
satisfies entirely the given differential system.

As a numerical example, let $f(x)$ and $g(x)$ be given by the graphs of Figure 49 below, where it is assumed that $L = 1$. Then

$$\begin{aligned} 100c_n &= \frac{2}{n\pi a} \int_0^{1/2} (-x) \sin n\pi x dx + \frac{2}{n\pi a} \int_{1/2}^1 (x-1) \sin n\pi x dx \\ &= -\frac{4(-1)^{(n-1)/2}}{n^3 \pi^3 a} \quad \text{if } n \text{ is odd} \\ &= 0 \quad \text{if } n \text{ is even,} \end{aligned}$$



(a)



(b)

Fig. 49

and

$$\begin{aligned}
 100b_n &= 2 \int_0^{1/2} (x) \sin n\pi x \, dx + 2 \int_{1/2}^1 (1-x) \sin n\pi x \, dx \\
 &= \frac{4(-1)^{(n-1)/2}}{n^3\pi^3} \quad \text{if } n \text{ is odd} \\
 &= 0 \quad \text{if } n \text{ is even.}
 \end{aligned}$$

By virtue of these formulas, Equation (144) gives

$$u(x,t) = \frac{1}{25\pi^3} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3} (-1)^{(n-1)/2} \sin n\pi x \left(\cos n\pi at - \frac{1}{n\pi a} \sin n\pi at \right). \quad (145)$$

Another method of treating nonhomogeneous boundary conditions is that of *superposition* of solutions. Suppose, for concreteness, we have the partial differential equation of Equation (141). Let $\phi(x,t)$ be the solution to this equation with the boundary conditions

1. $\phi(0,t) = 0,$
2. $\phi(L,t) = 0,$
3. $\frac{\partial}{\partial t} \phi(x,0) = 0,$
4. $\phi(x,0) = f(x).$

This problem has been solved completely in Section 42. Also, let $\psi(x,t)$ be the solution to Equation (141) with the boundary conditions

1. $\psi(0,t) = 0,$
2. $\psi(L,t) = 0,$
3. $\frac{\partial}{\partial t} \psi(x,0) = g(x),$
4. $\psi(x,0) = 0.$

This differential system can also be solved by the methods of Section 42. Now, clearly the function

$$u(x,t) = \phi(x,t) + \psi(x,t)$$

satisfies Equation (141) and the boundary conditions of Equation (142). First $u(x,t)$ satisfies the differential equation, since both ϕ and ψ are solutions. Secondly,

$$1. \quad u(0,t) = \phi(0,t) + \psi(0,t) = 0,$$

since both $\phi(0,t)$ and $\psi(0,t)$ are zero by hypothesis.

$$2. \quad u(L,t) = \phi(L,t) + \psi(L,t) = 0,$$

since again $\phi(L,t)$ and $\psi(L,t)$ are zero by hypothesis. Also,

$$3. \quad \frac{\partial}{\partial t} u(x,0) = \frac{\partial}{\partial t} \phi(x,0) + \frac{\partial}{\partial t} \psi(x,0) = g(x),$$

since $\partial\phi(x,0)/\partial t = 0$ and $\partial\psi(x,0)/\partial t = g(x)$. Finally,

$$4. \quad u(x,0) = \phi(x,0) + \psi(x,0) = f(x)$$

since $\phi(x,0) = f(x)$ and $\psi(x,0) = 0$.

50. Temperature distribution in a semicircular plate

The problems of the previous sections were all governed by partial differential equations with constant coefficients, and hence yielded ordinary differential equations with constant coefficients. In this section we shall consider an example which leads to an equation with variable coefficients (the so-called Cauchy equation). Other variable coefficient equations will be studied in Chapter V.

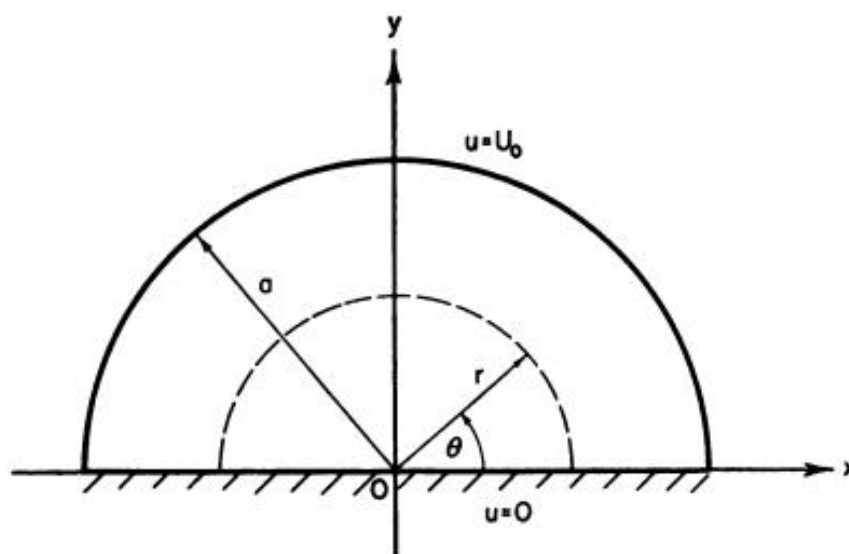


Fig. 50

We wish to find steady-state temperature distribution in a semicircular plate of radius a , insulated on both faces, with its curved boundary kept at a constant temperature U_0 and its bounding diameter kept at zero temperature (see Figure 50). The partial differential equation governing this phenomenon was found in Section 8 to be the Laplacian equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Since the cartesian form of Laplace's equation is not well adapted to deal with the problem in hand, we shall transform it into its polar form.

The relation between the cartesian coordinates x , y , and the polar coordinates r , θ is

$$x = r \cos \theta, \quad y = r \sin \theta,$$

or, inversely,

$$r = \sqrt{x^2 + y^2}, \quad \theta = \arctan \frac{y}{x}.$$

We must compute $\partial^2 u / \partial x^2$ and $\partial^2 u / \partial y^2$. Towards this end we find

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x}{r}, & \frac{\partial^2 r}{\partial x^2} &= \frac{1}{r} - \frac{x^2}{r^3}, \\ \frac{\partial \theta}{\partial x} &= -\frac{y}{r^2}, & \frac{\partial^2 \theta}{\partial x^2} &= \frac{2xy}{r^4}, \end{aligned}$$

and since

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta} \frac{\partial \theta}{\partial x}, \\ \frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial r} \frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 u}{\partial r^2} \left(\frac{\partial r}{\partial x} \right)^2 + \frac{\partial u}{\partial \theta} \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 u}{\partial \theta^2} \left(\frac{\partial \theta}{\partial x} \right)^2 \\ &\quad + 2 \frac{\partial r}{\partial x} \frac{\partial \theta}{\partial x} \left(\frac{\partial^2 u}{\partial \theta \partial r} \right), \end{aligned}$$

we have

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial r} \left(\frac{1}{r} - \frac{x^2}{r^3} \right) + \frac{\partial^2 u}{\partial r^2} \left(\frac{x}{r} \right)^2 + \frac{\partial u}{\partial \theta} \left(\frac{2xy}{r^4} \right) + \frac{\partial^2 u}{\partial \theta^2} \left(\frac{y^2}{r^4} \right) \\ &\quad - 2 \frac{\partial^2 u}{\partial \theta \partial r} \left(\frac{xy}{r^3} \right). \end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial r} \left(\frac{1}{r} - \frac{y^2}{r^3} \right) + \frac{\partial^2 u}{\partial r^2} \left(\frac{y}{r} \right)^2 + \frac{\partial u}{\partial \theta} \left(-\frac{2xy}{r^4} \right) + \frac{\partial^2 u}{\partial \theta^2} \left(\frac{x^2}{r^4} \right) \\ + 2 \frac{\partial^2 u}{\partial \theta \partial r} \left(\frac{yx}{r^3} \right).\end{aligned}$$

Therefore Laplace's equation $\nabla^2 u$ becomes in polar coordinates

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 u}{\partial r^2} \left(\frac{x^2 + y^2}{r^2} \right) + \frac{\partial^2 u}{\partial \theta^2} \left(\frac{x^2 + y^2}{r^4} \right) \\ &\quad + \frac{\partial u}{\partial r} \left(\frac{2}{r} - \frac{x^2 + y^2}{r^3} \right) + \frac{\partial u}{\partial \theta} \left(\frac{2xy}{r^4} - \frac{2xy}{r^4} \right) \\ &\quad + 2 \frac{\partial^2 u}{\partial \theta \partial r} \left(\frac{xy}{r^3} - \frac{xy}{r^3} \right) \\ &= \frac{1}{r^2} \left(r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} \right),\end{aligned}$$

and $\nabla^2 u = 0$ is therefore

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \quad (146)$$

in polar coordinates.

The temperature u considered as a function of r and θ (see Figure 50) satisfies Equation (146) and the boundary conditions

1. $u(a, \theta) = U_0$
2. $u(r, 0) = 0$
3. $u(r, \pi) = 0$.

The method of separation of variables suggests assuming a solution of Equation (146) of the form

$$u(r, \theta) = R(r) \Theta(\theta).$$

Substituting this into Equation (146) reduces it to

$$r^2 R'' \Theta + r R' \Theta + R \Theta'' = 0.$$

Dividing by $R \Theta$ and separating the variables, we obtain

$$r^2 \frac{R''}{R} + r \frac{R'}{R} = - \frac{\Theta''}{\Theta}. \quad (147)$$

Since the right-hand side is a function of θ only and the left-hand side a function of r only, their common value must be a constant, say λ . Since for $\lambda < 0$, the solutions of $\Theta'' + \lambda\Theta = 0$ are exponential in character, which is not physically reasonable, we assume $\lambda = \beta^2 > 0$. Equation (147) then yields the two ordinary differential equations

$$\begin{aligned} r^2 R'' + rR' - \beta^2 R &= 0, \\ \Theta'' + \beta^2 \Theta &= 0. \end{aligned} \quad (148)$$

The general solution of the second is

$$\Theta(\theta) = A \cos \beta\theta + B \sin \beta\theta. \quad (149)$$

The first equation is a so-called *Cauchy equation*. In order to solve this equation we make the transformation

$$r = e^s.$$

Then

$$\begin{aligned} R' &= \frac{dR}{dr} = \frac{dR}{ds} \frac{ds}{dr} = \frac{1}{r} \frac{dR}{ds}, \\ R'' &= \frac{1}{r^2} \frac{d^2 R}{ds^2} - \frac{1}{r^3} \frac{dR}{ds}, \end{aligned}$$

and the first of Equations (148) reduces to

$$\frac{d^2 R}{ds^2} - \beta^2 R = 0.$$

The general solution of this equation is

$$R = Ce^{\beta s} + De^{-\beta s},$$

or since $r = e^s$,

$$R = Cr^\beta + Dr^{-\beta}. \quad (150)$$

The arbitrary constants appearing in Equations (149) and (150) are determined by the boundary conditions. From the second boundary condition,

$$u(r, 0) = R(r) \Theta(0) = 0,$$

we conclude that $\Theta(0) = 0$ and hence that $A = 0$. From the third boundary condition,

$$u(r, \pi) = R(r) \Theta(\pi) = 0,$$

we conclude that $\Theta(\pi) = 0$, or

$$\Theta(\pi) = B \sin \beta\pi = 0.$$

Since $B \neq 0$ [otherwise $u(r, \theta) \equiv 0$] we must have $\sin \beta\pi = 0$, from which

$$\beta\pi = n\pi \quad \text{or} \quad \beta = n, \quad n = 1, 2, 3, \dots$$

Hence we find the solutions

$$\Theta_n(\theta) = B_n \sin n\theta.$$

Examining Equation (150) we see that as $r \rightarrow 0$, the term $r^{-\beta} \rightarrow \infty$ and hence makes $u(r, \theta)$ arbitrarily large for r very small. This contradicts the physical problem, since at $r = 0$, $u(r, \theta) = 0$. Therefore D must be equal to zero. The functions

$$u_n(r, \theta) = b_n r^n \sin n\theta$$

satisfy the partial differential equation and the last two boundary conditions. As before, any linear combination

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \quad (151)$$

also has this property whatever the values of the constants b_n . The first boundary condition is also satisfied if

$$u(a, \theta) = U_0 = \sum_{n=1}^{\infty} b_n a^n \sin n\theta,$$

that is, the $a^n b_n$ are the sine Fourier coefficients of U_0 ,

$$\begin{aligned} a^n b_n &= \frac{2}{\pi} \int_0^{\pi} U_0 \sin n\theta \, d\theta \\ &= \frac{4U_0}{n\pi} \quad \text{if } n \text{ is odd,} \\ &= 0 \quad \text{if } n \text{ is even.} \end{aligned}$$

Hence

$$b_n = \frac{4U_0}{a^n \pi n}$$

if n is odd, and is zero otherwise. The series of Equation (151) then becomes

$$u(r, \theta) = \frac{4U_0}{\pi} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n} \left(\frac{r}{a}\right)^n \sin n\theta, \quad (152)$$

which is the desired solution.

51. Transverse vibrations of a beam

As an example of an equation of order higher than the second, consider the transverse vibrations of a simply supported beam which has zero initial velocity and whose initial displacement is $f(x) = Ax(L - x)$, where A is a constant and L the length of the beam. The equation governing this phenomenon was found to be (Section 6)

$$\frac{\partial^2 y}{\partial t^2} + a^2 \frac{\partial^4 y}{\partial x^4} = 0 \quad \left(a^2 = \frac{EIg}{w}\right), \quad (153)$$

say with the boundary conditions :

$$\begin{aligned} 1. \quad & y(0, t) = 0, \\ 2. \quad & y(L, t) = 0, \\ 3. \quad & \frac{\partial^2}{\partial x^2} y(0, t) = 0, \\ 4. \quad & \frac{\partial^2}{\partial x^2} y(L, t) = 0, \\ 5. \quad & \frac{\partial}{\partial t} y(x, 0) = 0, \\ 6. \quad & y(x, 0) = Ax(L - x). \end{aligned} \quad (154)$$

Assume a solution of the form

$$y(x, t) = X(x) T(t),$$

which transforms Equation (153) into the equation

$$XT'' + a^2 X^{(4)} T = 0, \quad \text{or} \quad \frac{X^{(4)}}{X} = -\frac{T''}{a^2 T}.$$

The constant value of the two sides of this equation must be assumed positive from physical considerations, namely, to make $T(t)$ oscillatory. If we call it β^4 , we obtain the two ordinary differential equations

$$X'' - \beta^4 X = 0, \quad T'' + \alpha^2 \beta^4 T = 0.$$

The general solution of the X equation is

$$X(x) = A \sinh \beta x + B \cosh \beta x + C \sin \beta x + D \cos \beta x. \quad (155)$$

The first boundary condition,

$$y(0,t) = X(0) T(t) = 0,$$

implies that $X(0) = 0$ and hence that

$$X(0) = B + D = 0. \quad (156a)$$

The second boundary condition,

$$y(L,t) = X(L) T(t) = 0,$$

implies that $X(L) = 0$, and hence that

$$X(L) = A \sinh \beta L + B \cosh \beta L + C \sin \beta L + D \cos \beta L = 0. \quad (156b)$$

To utilize the third and fourth boundary conditions, we calculate $X''(x)$ from Equation (155):

$$X''(x) = A\beta^2 \sinh \beta x + B\beta^2 \cosh \beta x - C\beta^2 \sin \beta x - D\beta^2 \cos \beta x,$$

and find that the third boundary condition,

$$\frac{\partial^2}{\partial x^2} y(0,t) = X''(0) T(t) = 0,$$

requires that $X''(0)$ vanish. Hence

$$X''(0) = B\beta^2 - D\beta^2 = 0. \quad (156c)$$

The fourth boundary condition,

$$\frac{\partial^2}{\partial x^2} y(L,t) = X''(L) T(t) = 0,$$

requires that $X''(L)$ vanish, and hence

$$\begin{aligned} X''(L) &= A\beta^2 \sinh \beta L + B\beta^2 \cosh \beta L \\ &\quad - C\beta^2 \sin \beta L - D\beta^2 \cos \beta L = 0. \end{aligned} \quad (156d)$$

Equations (156a,b,c,d) are four linear algebraic equations on the constants A , B , C , D , and β . From Equations (156a,c) we have

$$B + D = 0, \quad B - D = 0,$$

and hence

$$B = D = 0.$$

From Equations (156b,d):

$$A \sinh \beta L + C \sin \beta L = 0,$$

$$A \sinh \beta L - C \sin \beta L = 0,$$

and hence

$$A \sinh \beta L = 0, \quad C \sin \beta L = 0.$$

The first condition gives $A = 0$ [since $\sinh \beta L = 0$ implies $\beta L = 0$, that is, $\beta = 0$ and hence $X(x) \equiv 0$]. If C is chosen equal to zero, then $X(x)$ is identically zero. Hence we assume that $\sin \beta L = 0$, or

$$\beta L = n\pi, \quad n = 1, 2, 3, \dots$$

and obtain the solutions

$$X_n(x) = C_n \sin \frac{n\pi}{L}x.$$

The ordinary differential equation

$$T'' + a^2\beta^2 T = 0$$

has the general solution

$$T(t) = E \cos a\beta^2 t + F \sin a\beta^2 t.$$

The fifth boundary condition,

$$\frac{\partial}{\partial t} y(x,0) = X(x) T'(0) = 0,$$

requires that $T'(0) = 0$, that is, that

$$T'(0) = -a\beta^2 E \sin a\beta^2 0 + a\beta^2 F \cos a\beta^2 0 = a\beta^2 F = 0.$$

Hence $F = 0$ and the solutions $T_n(t)$ become

$$T_n(t) = E_n \cos a \frac{n^2\pi^2}{L^2} t.$$

Therefore any function $y_n(x, t)$,

$$y_n(x, t) = X_n(x) T_n(t) = b_n \sin \frac{n\pi}{L}x \cos a \frac{n^2\pi^2}{L^2}t,$$

where $b_n = C_n E_n$ satisfies the partial differential equation of Equation (153) and the first five boundary conditions of Equation (154), whatever the values of the constants b_n . The linear combination of y_n ,

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L}x \cos a \frac{n^2\pi^2}{L^2}t,$$

also has this property, and will satisfy the sixth boundary condition if

$$y(x, 0) = Ax(L - x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L}x.$$

Hence the b_n are the sine Fourier coefficients of $Ax(L - x)$:

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L Ax(L - x) \sin \frac{n\pi}{L}x \, dx \\ &= \frac{8AL^2}{n^3\pi^3} \quad \text{if } n \text{ is odd,} \\ &= 0 \quad \text{if } n \text{ is even.} \end{aligned}$$

The solution to the boundary value problem of Equations (153) and (154) is therefore

$$y(x, t) = \frac{8AL^2}{\pi^3} \sum_{n=1,3,5,\dots}^{\infty} \frac{1}{n^3} \sin \frac{n\pi}{L}x \cos a \frac{n^2\pi^2}{L^2}t. \quad (157)$$

52. The vibrating membrane

As an example of a problem involving *three* independent variables, consider the equation of the vibrating membrane (Section 4)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}. \quad (158)$$

We shall assume that the membrane is rectangular in shape (see Figure 51) and satisfies the following boundary conditions :

1. $u(0, y, t) = 0,$
2. $u(a, y, t) = 0,$
3. $u(x, 0, t) = 0,$
4. $u(x, b, t) = 0,$
5. $\frac{\partial}{\partial t} u(x, y, 0) = 0,$
6. $u(x, y, 0) = Txy(x - a)(y - b).$

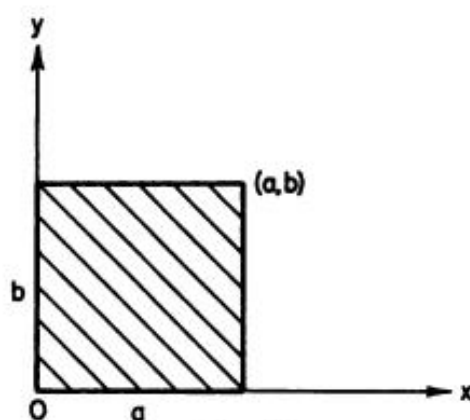


Fig. 51

The first four boundary conditions state that the membrane is held fixed on the boundary for all time. The fifth states that the membrane has no initial velocity, while the sixth gives the initial displacement.

As customary in the method of separation of variables, assume a solution of the form

$$u(x, y, t) = X(x) Y(y) T(t).$$

Substituting it in Equation (158) and dividing by XYT , we obtain

$$\frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T} - \frac{Y''}{Y}. \quad (159)$$

The right-hand side of this equation is independent of x , and the left-hand side is independent of t and y . Their common value must be a constant λ . Further, from physical considerations, λ must be negative, say $\lambda = -\beta^2$, for otherwise $X(x)$ would vary

exponentially and the boundary conditions could not be satisfied. Equation (159) thus gives the two separate equations :

$$X'' + \beta^2 X = 0 \quad (160)$$

and

$$\frac{Y''}{Y} = \frac{T''}{c^2 T} + \beta^2. \quad (161)$$

In Equation (161) the right-hand side is independent of y , while the left-hand side is independent of t ; hence their common value must be a constant, say μ . As before, we let $\mu = -\alpha^2$, and hence Equation (161) gives the two separate equations

$$Y'' + \alpha^2 Y = 0 \quad (162)$$

and

$$T'' + c^2(\alpha^2 + \beta^2)T = 0. \quad (163)$$

The partial differential equation of Equation (158) has thus been replaced by *three* ordinary differential equations, Equations (160), (162), and (163). The solutions of these three equations are, respectively,

$$\begin{aligned} X(x) &= A \cos \beta x + B \sin \beta x, \\ Y(y) &= C \cos \alpha y + D \sin \alpha y, \\ T(t) &= E \cos \gamma t + F \sin \gamma t, \end{aligned} \quad (164)$$

where

$$\gamma^2 = c^2(\alpha^2 + \beta^2).$$

From the first boundary condition,

$$u(0, y, t) = X(0) Y(y) T(t) = 0,$$

we conclude that $X(0) = 0$ and hence that $A = 0$. From the second,

$$u(a, y, t) = X(a) Y(y) T(t) = 0,$$

we conclude that $X(a) = 0$ and hence that $B \sin \beta a = 0$. If u is not to be identically zero, we must have $\sin \beta a = 0$, or

$$\beta a = n\pi, \quad n = 1, 2, 3, \dots \quad (165)$$

Similarly, the third and fourth boundary conditions imply that $C = 0$ and that

$$\alpha b = m\pi, \quad m = 1, 2, 3, \dots \quad (166)$$

Finally, from the fifth boundary condition,

$$\frac{\partial}{\partial t} u(x, y, 0) = X(x) Y(y) T'(0) = 0,$$

we conclude that

$$T'(0) = -\gamma E \sin 0 + \gamma F \cos 0 = +\gamma F = 0$$

and hence that $F = 0$.

Indicating the value of BDE corresponding to the integers n and m in Equations (165) and (166) by a_{nm} , we may write

$$u_{nm}(x, y, t) = a_{nm} \sin \frac{n\pi}{a} x \sin \frac{m\pi}{b} y \cos \left[ct \sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2} \right].$$

These functions satisfy the partial differential equation and the first five boundary conditions for any choice of the constants a_{nm} and for any integral values of n and m . In order to satisfy the sixth boundary condition we assume for u the linear combination

$$u(x, y, t) = \sum_n \sum_m u_{nm}(x, y, t).$$

Letting $t = 0$ in this series, we see that

$$u(x, y, 0) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \sin \frac{n\pi}{a} x \sin \frac{m\pi}{b} y$$

and hence the a_{nm} are the sine Fourier coefficients of the function $u(x, y, 0)$. By Section 38,

$$\begin{aligned} a_{nm} &= \frac{4}{ab} \int_0^a \int_0^b u(x, y, 0) \sin \frac{n\pi}{a} x \sin \frac{m\pi}{b} y \, dx \, dy \\ &= \frac{4T}{ab} \int_0^a x(x-a) \sin \frac{n\pi}{a} x \, dx \int_0^b y(y-b) \sin \frac{m\pi}{b} y \, dy \\ &= \frac{4T}{ab} \frac{2a^3}{\pi^3 n^3} [(-1)^n - 1] \frac{2b^3}{\pi^3 m^3} [(-1)^m - 1] \\ &= \frac{64Ta^3b^3}{\pi^6 m^3 n^3} \quad \text{if } n \text{ and } m \text{ are odd,} \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Our final solution is therefore

$$u(x, y, t) = \frac{64Ta^3b^3}{\pi^6} \sum_{n=1,3,5,\dots}^{\infty} \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{n^3m^3} \sin \frac{n\pi}{a}x \sin \frac{m\pi}{b}y \cos \gamma_{nm}t, \quad (167)$$

where

$$\gamma_{nm}^2 = \left[\left(\frac{n\pi}{a} \right)^2 + \left(\frac{m\pi}{b} \right)^2 \right] c^2.$$

53. Nonhomogeneous equations

The partial differential equations considered in the previous examples were all homogeneous. In this section we shall consider a certain class of nonhomogeneous equations, namely, those in which the function on the right-hand side of the equation is a function of *one* variable only (the so-called forcing function). Many physical problems give rise to equations that fit into this category, of which a few will be mentioned.

1. *Radioactive Decay.* We have seen in Section 7 that the temperature distribution in a bar of length L satisfies the one-dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2} - a^2 \frac{\partial u}{\partial t} = 0.$$

If now we assume that heat is also gained by radioactive decay of the material in the bar (see Problem 20, Chapter I) an additional term of the form Ae^{-ax} must be added, and the above equation modified to read

$$\frac{\partial^2 u}{\partial x^2} - a^2 \frac{\partial u}{\partial t} = -Ae^{-ax}. \quad (168)$$

Typical boundary conditions associated with this equation might be

$$\begin{aligned} 1. \quad & u(0, t) = 0, \\ 2. \quad & u(L, t) = 0, \\ 3. \quad & u(x, 0) = f(x), \end{aligned} \quad (169)$$

where $f(x)$ is a given function of x .

2. *Decomposition.* Consider again the one-dimensional flow of heat and assume that heat is also gained by the decomposition of the material in the bar (see Problem 21, Chapter I). This heat is generally assumed to be generated at a constant rate A , and the heat equation therefore assumes the form

$$\frac{\partial^2 u}{\partial x^2} - a^2 \frac{\partial u}{\partial t} = -A. \quad (170)$$

Appropriate boundary conditions might be

$$\begin{aligned} 1. \quad u(0,t) &= 0, \\ 2. \quad u(L,t) &= 0, \\ 3. \quad u(x,0) &= 0. \end{aligned} \quad (171)$$

3. *Forced Vibrations.* Consider the equation of the vibrating string,

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2},$$

and assume that an external force normal to the string is applied which varies, say linearly, with the distance from the end $x = 0$ (see Problem 22, Chapter I). In this case the equation assumes the form

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = Ax. \quad (172)$$

Boundary conditions for such a problem might be

$$\begin{aligned} 1. \quad u(0,t) &= 0, \\ 2. \quad u(L,t) &= 0, \\ 3. \quad u(x,0) &= 0, \\ 4. \quad \frac{\partial}{\partial t} u(x,0) &= 0. \end{aligned} \quad (173)$$

Note that here, as in the second example, all the boundary conditions are homogeneous, yet there exists a nontrivial solution, since the differential equation is nonhomogeneous. [If both the equation and the boundary conditions are homogeneous, the only solution is the trivial one, $u(x,t) = 0$.]

The technique used in solving equations of the type given above consists in introducing a new function of the form

$w(x,t) + \psi(x)$, where $w(x,t)$ satisfies a *homogeneous* partial differential equation and $\psi(x)$ is determined from the solution of an *ordinary* differential equation. We shall illustrate this method in the next section.

54. The heat equation with radioactive decay

To start the solution of Equation (168) we introduce a new function $w(x,t)$ defined by the equation

$$u(x,t) = w(x,t) + \psi(x), \quad (174)$$

where $\psi(x)$ is a function of x only, as yet undetermined. We then try to determine $\psi(x)$ so that $w(x,t)$ satisfies the *homogeneous* equation obtained from Equation (168) by setting $A = 0$. Substituting Equation (174) into Equation (168) we obtain

$$\frac{\partial^2 w}{\partial x^2} + \psi''(x) - a^2 \frac{\partial w}{\partial t} = -Ae^{-ax}.$$

Hence, if $\psi(x)$ is such that

$$\psi''(x) = -Ae^{-ax}, \quad (175)$$

$w(x,t)$ will satisfy the homogeneous equation

$$\frac{\partial^2 w}{\partial x^2} - a^2 \frac{\partial w}{\partial t} = 0.$$

Equation (175) is readily integrated and gives

$$\psi(x) = -\frac{A}{a^2}e^{-ax} + Bx + C, \quad (176)$$

where B and C are arbitrary constants. Now any value may be used for B and C , and $w(x,t)$ will still satisfy a homogeneous equation. At first we might be tempted to let B and C be zero. A little reflection will indicate, that, since we have the arbitrariness of B and C at our disposal, it might be well to determine them such that the boundary conditions assume as simple a form as possible. From the first of Equations (169) and Equation (174) we see that

$$u(0,t) = w(0,t) + \psi(0) = 0,$$

or

$$w(0,t) = -\psi(0).$$

Similarly, from the second boundary condition, we obtain

$$u(L,t) = w(L,t) + \psi(L) = 0,$$

or

$$w(L,t) = -\psi(L).$$

The constants B and C will be so chosen that

$$\psi(0) = 0 = \psi(L).$$

Thus we obtain from Equation (176)

$$\psi(0) = -\frac{A}{\alpha^2} + C = 0, \quad \text{or} \quad C = \frac{A}{\alpha^2}.$$

Also,

$$\psi(L) = -\frac{A}{\alpha^2}e^{-\alpha L} + BL + \frac{A}{\alpha^2} = 0,$$

or

$$B = \frac{1}{L} \left(\frac{A}{\alpha^2}e^{-\alpha L} - \frac{A}{\alpha^2} \right). \quad (177)$$

With these values of B and C , the function $w(x,t)$ satisfies the homogeneous equation

$$\frac{\partial^2 w}{\partial x^2} - a^2 \frac{\partial w}{\partial t} = 0$$

and the boundary conditions

1. $w(0,t) = 0,$
2. $w(L,t) = 0,$
3. $w(x,0) = f(x) - \psi(x)$

$$= f(x) + \frac{A}{\alpha^2} \left[e^{-\alpha x} - \frac{x}{L}(e^{-\alpha L} - 1) - 1 \right].$$

Once this system is solved by the techniques developed in previous sections, the original function $u(x,t)$ is readily determined from the equation

$$\begin{aligned} u(x,t) &= w(x,t) + \psi(x) \\ &= w(x,t) + \frac{A}{\alpha^2} \left[-e^{-\alpha x} + \frac{x}{L}(e^{-\alpha L} - 1) + 1 \right]. \end{aligned} \quad (178)$$

55. The general case

A rather wide class of homogeneous linear partial differential equations of the second order in two independent variables is solvable by the method of separation of variables—namely, the class of equations which can be put into the form :

$$\begin{aligned} f_1(x) \frac{\partial^2 u}{\partial x^2} + \phi_1(y) \frac{\partial^2 u}{\partial y^2} + f_2(x) \frac{\partial u}{\partial x} \\ + \phi_2(y) \frac{\partial u}{\partial y} + [f_3(x) + \phi_3(y)]u = 0. \end{aligned} \quad (179)$$

This family includes as special cases many of the equations considered thus far. If we assume a solution of the form

$$u(x, y) = X(x) Y(y)$$

and substitute it in Equation (179), we obtain

$$f_1 X'' Y + \phi_1 X Y'' + f_2 X' Y + \phi_2 X Y' + (f_3 + \phi_3) X Y = 0,$$

or

$$Y[f_1 X'' + f_2 X' + f_3 X] + X[\phi_1 Y'' + \phi_2 Y' + \phi_3 Y] = 0.$$

Dividing through by XY , this equation reduces by the usual arguments to the two ordinary differential equations

$$\begin{aligned} f_1 X'' + f_2 X' + f_3 X &= \lambda X, \\ \phi_1 Y'' + \phi_2 Y' + \phi_3 Y &= -\lambda Y, \end{aligned}$$

where λ is an undetermined constant. The solution of these ordinary differential equations is, in general, not a trivial problem, but the separation of the variables has been achieved.

56. A trial function

Another elementary method frequently useful in solving partial differential equations is to assume a solution of a particular form involving a number of arbitrary constants and/or functions and then try to determine these constants and/or functions so as to satisfy the given differential system. We shall illustrate this method by means of an example. Consider first the familiar equation of the vibrating string,

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \quad (180)$$

with the boundary conditions

1. $u(0,t) = 0,$
2. $u(L,t) = 0,$
3. $u(x,0) = D \sin \frac{\pi}{L}x,$
4. $\frac{\partial}{\partial t} u(x,0) = 0.$

A trial function involving sines and cosines seems appropriate since only second derivatives are involved in Equation (180);

$$(d^2 \sin x)/dx^2 = -\sin x, \quad (d^2 \cos x)/dx^2 = -\cos x;$$

and furthermore the boundary conditions involve a trigonometric function. Let us therefore assume a solution of the form

$$u(x,t) = A \sin \alpha x \cos \beta t, \quad (181)$$

where A , α , and β are constants as yet unspecified. Substitution of Equation (181) into Equation (180) yields

$$-a^2 \alpha^2 A \sin \alpha x \cos \beta t = -\beta^2 A \sin \alpha x \cos \beta t,$$

and shows that Equation (181) is a solution of Equation (180) if

$$a^2 \alpha^2 = \beta^2,$$

that is, if

$$u(x,t) = A \sin \alpha x \cos a\alpha t,$$

where A and α are still arbitrary. This function can be made to satisfy the boundary conditions by choosing A and α properly. In fact, the first boundary condition is automatically satisfied,

$$u(0,t) = A \sin 0 \cos a\alpha t = 0.$$

The second condition gives

$$u(L,t) = A \sin \alpha L \cos a\alpha t = 0,$$

or

$$\alpha = \frac{n\pi}{L} \quad n = 1, 2, \dots$$

The third boundary condition yields the equation

$$u(x,0) = A \sin \frac{n\pi}{L}x = D \sin \frac{\pi}{L}x,$$

which becomes an identity if

$$n = 1 \quad \text{and} \quad A = D.$$

Hence

$$u(x,t) = D \sin \frac{\pi}{L}x \cos \frac{\alpha\pi}{L}t \quad (182)$$

is the desired solution, since it satisfies entirely the differential system (the fourth boundary condition being automatically satisfied). In general, if a more complicated trial solution than Equation (181) had been assumed, the additional terms would have dropped out.

57. A trial function method based on Fourier series

A variation of the trial function method consists in assuming the solution of a partial differential system can be written in the form of a Fourier series. This method might be called the "direct Fourier series method" and is analogous to the methods used in solving ordinary differential equations by power series. To illustrate this approach we shall consider the heat equation in two dimensions. If we have a two dimensional plate, the equation satisfied by the temperature u was found to be (Section 8)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \frac{\partial u}{\partial t}. \quad (183)$$

We shall assume that the plate (see Figure 47) is insulated on its lateral surfaces, has its four edges kept at zero temperature, and has a temperature T (a constant) at $t = 0$. The boundary conditions for this problem may therefore be written

$$\begin{aligned} 1. \quad & u(0,y,t) = 0, \\ 2. \quad & u(a,y,t) = 0, \\ 3. \quad & u(x,0,t) = 0, \\ 4. \quad & u(x,b,t) = 0, \\ 5. \quad & u(x,y,0) = T. \end{aligned} \quad (184)$$

Let us prolong the function $u(x,y,0)$ as an *odd* function in both the x and y variables so that the temperature distribution $u(x,y,t)$

will also be an odd function of x and y for all time t . That is, we shall assume $u(x, y, t)$ can be written in the form

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} f_{nm}(t) \sin \frac{n\pi}{a}x \sin \frac{m\pi}{b}y, \quad (185)$$

where the coefficients f_{nm} of the double Fourier series are functions of time. We see that if $u(x, y, t)$ is so defined, it automatically identically satisfies the first four boundary conditions of Equations (184).

If we substitute a typical term of Equation (185),

$$f_{nm}(t) \sin \frac{n\pi}{a}x \sin \frac{m\pi}{b}y$$

into Equation (183), we have

$$\begin{aligned} -\left(\frac{n\pi}{a}\right)^2 f_{nm} \sin \frac{n\pi}{a}x \sin \frac{m\pi}{b}y - \left(\frac{m\pi}{b}\right)^2 f_{nm} \sin \frac{n\pi}{a}x \sin \frac{m\pi}{b}y \\ = \frac{1}{c^2} f'_{nm} \sin \frac{n\pi}{a}x \sin \frac{m\pi}{b}y. \end{aligned}$$

If this equation is to be identically satisfied we must have

$$-\left(\frac{n\pi}{a}\right)^2 f_{nm} - \left(\frac{m\pi}{b}\right)^2 f_{nm} = \frac{1}{c^2} f'_{nm},$$

or

$$f'_{nm}(t) + c^2 \beta_{nm}^2 f_{nm}(t) = 0, \quad (186)$$

where

$$\beta_{nm}^2 = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2.$$

The solution of Equation (186) is

$$f_{nm}(t) = A_{nm} e^{-c^2 \beta_{nm}^2 t},$$

and hence

$$u(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} e^{-c^2 \beta_{nm}^2 t} \sin \frac{n\pi}{a}x \sin \frac{m\pi}{b}y \quad (187)$$

is a function which satisfies the partial differential equation of Equation (183) and the first four boundary conditions of Equations (184).

tion (184). If u is to satisfy the fifth boundary condition, we must have

$$u(x, y, 0) = T = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{nm} \sin \frac{n\pi}{a} x \sin \frac{m\pi}{b} y.$$

Hence the A_{nm} are the Fourier sine coefficients of the double Fourier series of $u(x, y, 0)$:

$$\begin{aligned} A_{nm} &= \frac{4}{ab} \int_0^a \int_0^b T \sin \frac{n\pi}{a} x \sin \frac{m\pi}{b} y \, dx \, dy \\ &= \frac{16T}{nm\pi^2} \quad \text{if } n \text{ and } m \text{ are odd,} \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

The solution to the heat problem of Equations (183) and (184) is therefore

$$u(x, y, t) = \frac{16T}{\pi^2} \sum_{n=1,3,5,\dots}^{\infty} \sum_{m=1,3,5,\dots}^{\infty} \frac{1}{nm} \sin \frac{n\pi}{a} x \sin \frac{m\pi}{b} y e^{-c^2 \beta_{nm}^2 t}, \quad (188)$$

where

$$\beta_{nm}^2 = \left(\frac{n\pi}{a} \right)^2 + \left(\frac{m\pi}{b} \right)^2.$$

In conclusion we might say that while the trial function method by Fourier series is generally simpler than the method of separation of variables, it is of limited application, due to the fact that the assumed solution must automatically satisfy one or more boundary conditions.

PROBLEMS

1. Verify the values of the coefficients of the Fourier series in the example of the vibrating string, Section 44.

2. Figure 52 shows the initial displacement of a perfectly flexible string of weight w lb per ft, fixed at the end points and under a tension of T lb. At time $t = 0$, the string is released from this position,

(a) Write the expression for the deflection $u(x, t)$

(b) Assuming $T = 10$ lb, $w = 0.0322$ lb/ft, $g = 32.2$ ft/sec², $L = 10$ ft, $\alpha = 1/3$, plot the graph of $u(x, t)$ at $t = n/100$, where $n = 1, 2, 3, 4, 5$, dividing the interval $(0, L)$ into ten equal parts.

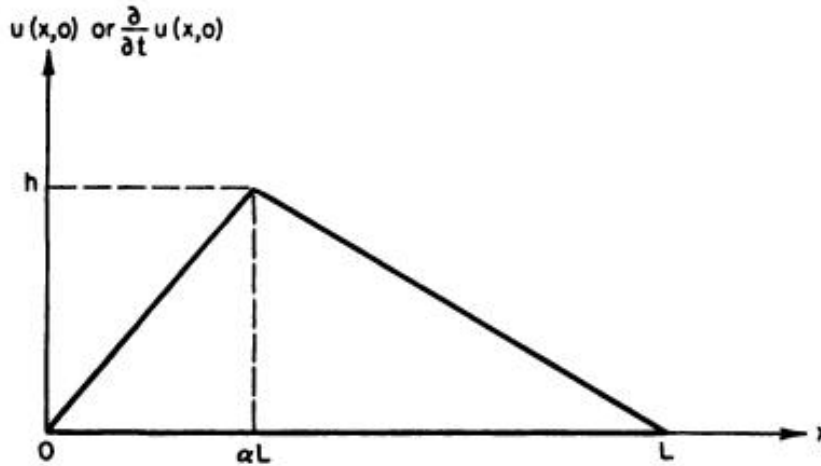


Fig. 52

3. Solve the problem of the vibrating string for the boundary conditions

1. $u(0, t) = 0$,
2. $u(L, t) = 0$,
3. $\frac{\partial}{\partial t} u(x, 0) = 0$,
4. $u(x, 0) = A(L \sinh x - x \sinh L)$.

(Use the constants T, w, g, L of Problem 2.)

4. Find the displacement $u(x, t)$ if the initial displacement of a vibrating string is zero, but the initial *velocity* is given by the function of Figure 52. Assume $\alpha = 0.5$.

5. Solve the problem of the vibrating string for the boundary conditions

1. $u(0, t) = 0$,
2. $u(L, t) = 0$,
3. $\frac{\partial}{\partial t} u(x, 0) = Ax(L - x)$,
4. $u(x, 0) \stackrel{!}{=} 0$.

6. Solve the problem of the vibrating string for the following boundary conditions :

1. $u(0, t) = 0,$
2. $u(1, t) = 0,$
3. $\frac{\partial}{\partial t} u(x, 0) = \frac{x(x-1)}{100},$
4. $u(x, 0) = \begin{cases} \frac{x}{100}, & 0 < x < \frac{1}{2} \\ \frac{1-x}{100}, & \frac{1}{2} < x < 1. \end{cases}$

Assume $a^2 = 2500$ and $L = 1$.

7. Solve the problem of the vibrating string if the string (of length L) is fixed at the end points, and has an initial velocity given by Figure 52 ($\alpha = \frac{1}{4}$) and an initial displacement given by Figure 53.

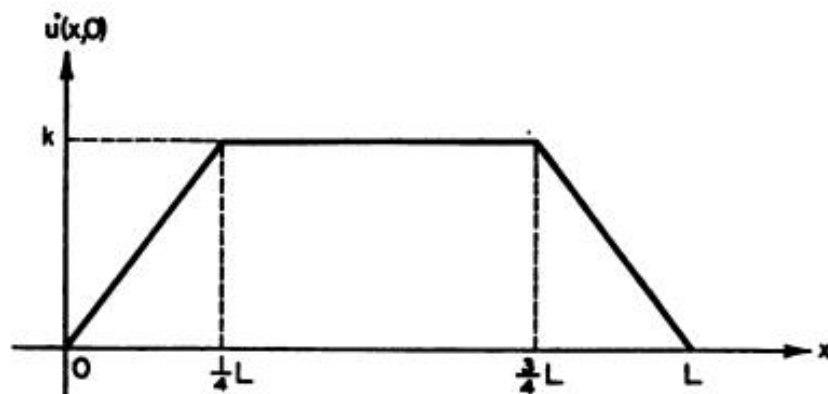


Fig. 53

8. Introduce in the equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

the change of variables

$$x = \xi + \eta, \quad y = \xi - \eta,$$

and obtain an explicit integral representation for $u(x, y)$ subject to the boundary conditions

$$u(x, 0) = f(x), \quad \frac{\partial}{\partial t} u(x, 0) = g(x).$$

9. Deduce the formula derived in Problem 8 by a consideration of the Fourier series of Equation (144), Section 49.

10. Solve Problem 3 of Chapter I.

11. We have seen in Problem 11 of Chapter I that the angular displacement θ of a shaft vibrating torsionally satisfies the equation

$$\frac{\partial^2 \theta}{\partial t^2} = a^2 \frac{\partial^2 \theta}{\partial x^2},$$

where $a^2 = Gg/\delta$, G being the shear modulus, δ the density, and g the gravitational constant. Find the twist $\theta(x, t)$ of a steel shaft 10 ft long, of specific gravity 8, and shear modulus 12×10^6 psi if it is fixed at both ends and given an initial twist $\theta(x, 0) = \frac{x(10 - x)}{800}$ and initial velocity zero.

12. Attempt the solution of Problem 17 of Chapter I for the longitudinal displacement of the bar.

13. Verify the values of the Fourier coefficients derived in solving the heat equation of Section 46.

14. Find the steady-state temperature distribution in a rectangular slab of sides a and b , insulated on the lateral surfaces and satisfying the boundary conditions

1. $u(0, y) = 0$,
2. $u(a, y) = 0$,
3. $u(x, b) = 0$,
4. $u(x, 0) = Tx(a - x)$.

15. Solve the steady-state heat equation for a slab in the form of a rectangle (see Figure 47) if the side along the x -axis has a temperature distribution of the form of Figure 53 ($L = a$), the remaining three sides are kept at zero temperature, and the lateral faces of the slab are insulated.

16. A rectangular plate of dimension a units along the x -axis and b units along the y -axis has two of its edges $x = 0$ and $x = a$ insulated. The edge $y = 0$ is kept at a constant temperature T , and the edge $y = b$ is kept at zero temperature. Find the steady-state temperature distribution in the plate. Assume that the lateral faces are insulated.

17. Find the steady-state temperature distribution $u(x, y)$ in a semi-infinite strip whose faces are insulated, whose edges $x = 0$ and $x = a$ are kept at zero temperature, and whose base $y = 0$ is kept at a constant temperature T .

18. Solve Problem 17 for the case in which the edges instead of being kept at zero temperature, are insulated.

19. Solve Problem 17 for the boundary conditions in which the edge $x = 0$ is kept at zero temperature, the edge $x = a$ is insulated, and the base is kept at a constant temperature T .

20. The constant λ in Section 45 was assumed negative. Try carrying out the solution if $\lambda = \alpha^2 > 0$.

21. A bar of length L has an initial temperature distribution as shown in Figure 54. From $t = 0$ on, the ends of the bar are kept at zero temperature. Find the equation for the temperature $u(x, t)$ of the bar. Assume the surface of the bar is insulated.

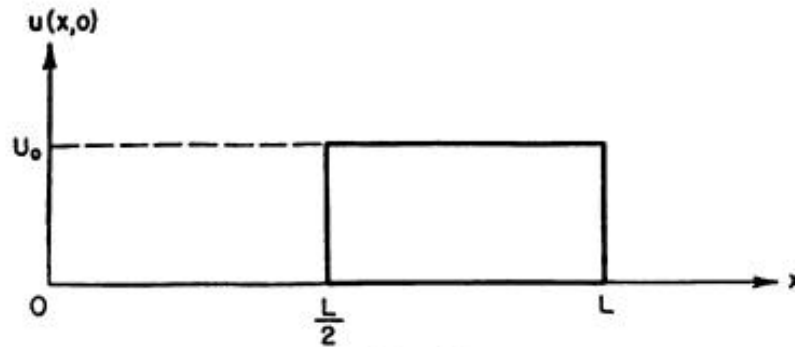


Fig. 54

22. Find the temperature distribution $u(x, t)$ in the bar of Problem 21 if it has an initial temperature distribution as given in Figure 53.

23. Solve the heat equation for one-dimensional transient flow with the boundary conditions

1. $u(0, t) = 0$,
2. $u(L, t) = 0$,
3. $u(x, 0) = x(L - x)$.

24. Solve the heat equation

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t}$$

subject to the boundary conditions

1. $u(0, t) = A$ (a constant),
2. $u(L, t) = B$ (a constant),
3. $u(x, 0) = x^2(L - x)$.

25. Solve the one-dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t}$$

with the adiabatic boundary conditions

$$1. \quad \frac{\partial}{\partial x} u(0, t) = 0,$$

$$2. \quad \frac{\partial}{\partial x} u(L, t) = 0,$$

$$3. \quad u(x, 0) = x.$$

26. Solve the submarine cable equations for the current and voltage in a cable of length l if at $x = 0$ and $x = l$ the cable is short-circuited, while the initial current distribution $i(x, 0)$ is $x(l - x)$.

27. A high-frequency transmission line of length l is grounded at $x = l$ and open-circuited at $x = 0$. If the initial current and voltage distributions are $i(x, 0) = I_0 \sin(7\pi/l)x$ and $e(x, 0) = E_0(l \sinh x - x \sinh l)$, respectively, determine the current and voltage in the line at any time t .

28. A semicircular plate of radius A has its circumference kept at a temperature $u(A, \theta) = T\theta(\pi - \theta)$, while the bounding diameter is kept at zero temperature. Find the temperature distribution $u(r, \theta)$. Assume the lateral surfaces of the disk to be insulated.

29. A ring-shaped plate (see Figure 55) of inner radius a and outer radius b is insulated on its lateral surfaces. Find the steady-state temperature $u(r, \theta)$ if the initial temperature distribution on the inner circle is $A\theta(2\pi - \theta)$, and the initial temperature distribution on the outer circle is $B\theta^2(2\pi - \theta)$.

30. Investigate the solution of Problem 29 as the inner radius a approaches zero.

31. A beam of length L is simply supported at $x = 0$, $x = L$, and is acted upon by two bending moments M_0 equal and opposite in sign at these end points. The deflection due to these moments is

$$\frac{M_0}{2EI} x(L - x).$$

At time $t = 0$, the moments are suddenly removed. Write the equation for the transverse motion $y(x, t)$ of the beam.

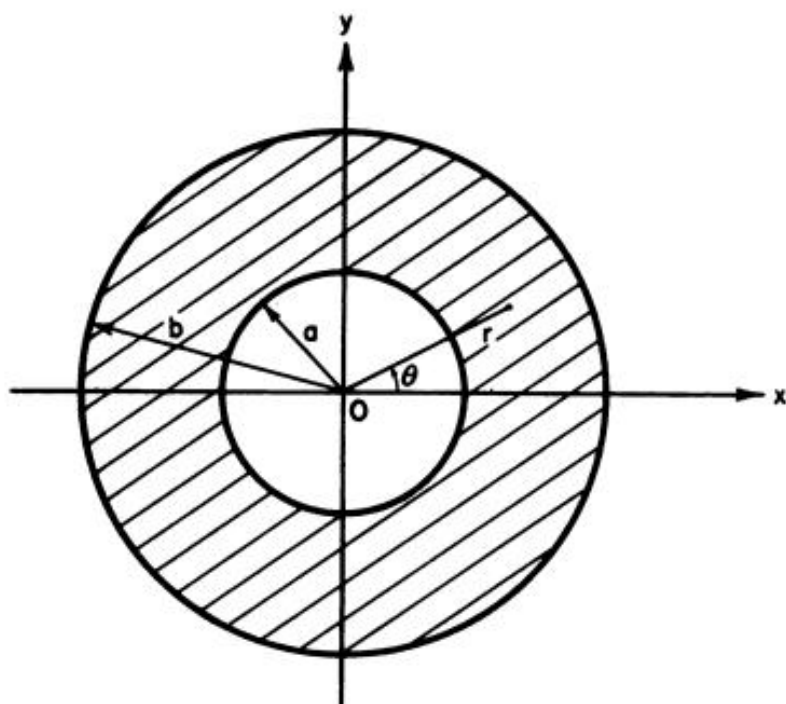


Fig. 55

32. A beam of length L is simply supported. If its initial velocity is zero and its initial displacement is $D_0 x \sin \pi x/L$, find the equation of its transverse vibrations.

33. Solve the example of Section 57 by the method of separation of variables.

34. Solve Problem 49, using the method of separation of variables.

35. Solve Problem 50, using the method of separation of variables.

36. Find the steady-state temperature $u(x, y, z)$ in a rectangular parallelepiped (see Figure 56), if the face in the xy -plane is kept at a temperature given by

$$u(x, y, 0) = U_0 xy(a - x)(b - y),$$

and the remaining faces are kept at zero temperature.

37. Solve the heat equation with radioactive decay,

$$\frac{\partial^2 u}{\partial x^2} - a^2 \frac{\partial u}{\partial t} + Ae^{-ax} = 0,$$

with the boundary conditions

1. $u(0, t) = 0$,
2. $u(L, t) = 0$,
3. $u(x, 0) = T$ (a constant).

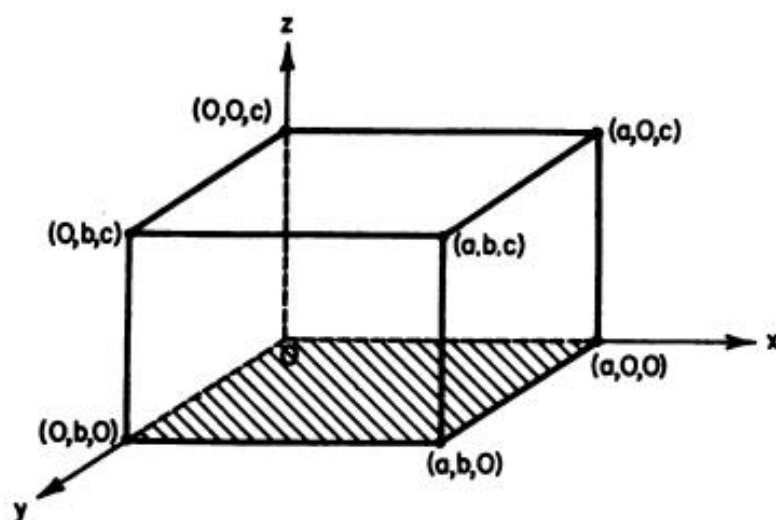


Fig. 56

38. Solve Problem 37 with completely homogeneous boundary conditions.

39. Solve the heat equation with decomposition,

$$\frac{\partial^2 u}{\partial x^2} - a^2 \frac{\partial u}{\partial t} + A = 0,$$

with the boundary conditions

1. $u(0,t) = 0,$
2. $u(L,t) = 0,$
3. $u(x,0) = 0.$

40. Determine the general form of the solution of the following boundary value problem :

$$\frac{\partial^2 u}{\partial x^2} = a^2 \frac{\partial u}{\partial t} + f(x) ;$$

1. $u(0,t) = a,$
2. $u(L,t) = b,$
3. $u(x,0) = g(x).$

41. Investigate the solution of Problem 40 where $f(x)$ is replaced by an arbitrary function of time, $\phi(t)$.

42. Solve the equation for the forced vibrations of a string,

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} + Ax,$$

with the boundary conditions

1. $u(0,t) = 0,$
2. $u(L,t) = 0,$
3. $u(x,0) = 0,$
4. $\frac{\partial}{\partial t} u(x,0) = 0.$

43. Solve Problem 19 of Chapter I for the temperature in the thin bar. Assume that the ends of the bar are kept at the temperature of the surroundings, T , and that the initial temperature $u(x, 0) = Te^{-x}$.

44. Solve the equation of the vibrating string (of length L) in which the initial velocity is given by

$$V_0 \sin^3 \frac{\pi}{L}x,$$

and the initial displacement is zero. If the string is 3 ft long, weighs 0.1 lb, is subjected to a constant tension of 5 lb, and $V_0 = 1.5$ fps, calculate the displacement of the mid-point of the string at a time 0.01 sec after the string was given its initial velocity.

45. Solve the equation of the vibrating string in which the initial displacement is given by

$$D_0 \left(\sin \frac{\pi}{L}x + 4 \sin \frac{5\pi}{L}x \right),$$

and the initial velocity is zero.

46. Solve the heat equation

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

for the boundary conditions

1. $u(0,t) = 0,$
2. $u(L,t) = 0,$
3. $u(x,0) = T \sin \frac{3\pi}{L}x.$

47. Solve Problem 28 of Chapter I by assuming a trial function in the form of a polynomial.

48. Solve the illustrative example of Section 52 by the trial function method with a Fourier series expansion.

49. Let a rectangular membrane, fixed on its boundary (Figure 51) have an initial displacement and an initial velocity of

$$u(x, y, 0) = Ax^2y(a - x)(b - y),$$
$$\frac{\partial}{\partial t} u(x, y, 0) = 0,$$

respectively. Find the coefficients a_{nm} of the double Fourier series expansion for $u(x, y, t)$, using the trial function method with a Fourier series expansion. Plot the displacement of the center of the membrane as a function of time for $a = 3$, $b = 2$, $A = 2$.

50. Solve Problem 49 with the boundary conditions

$$u(x, y, 0) = 0,$$
$$\frac{\partial}{\partial t} u(x, y, 0) = Axy^2(a - x)(b - y).$$

Chapter IV

THE FOURIER INTEGRAL

58. A heat flow problem

Consider an infinitely long thin bar insulated on its lateral surface (see Figure 57). We wish to determine the temperature $u(x,t)$ if the initial temperature is given by

$$u(x,0) = f(x) \quad -\infty < x < \infty. \quad (189)$$

The equation satisfied by $u(x,t)$ was shown in Section 7 to be

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad (190)$$

and this problem appears similar to those treated in Chapter III. There is, however, one important difference: the bar is *infinite* in extent. If we try to solve this problem by the method of separation of variables we shall see that Fourier series are not adequate to yield a complete solution.

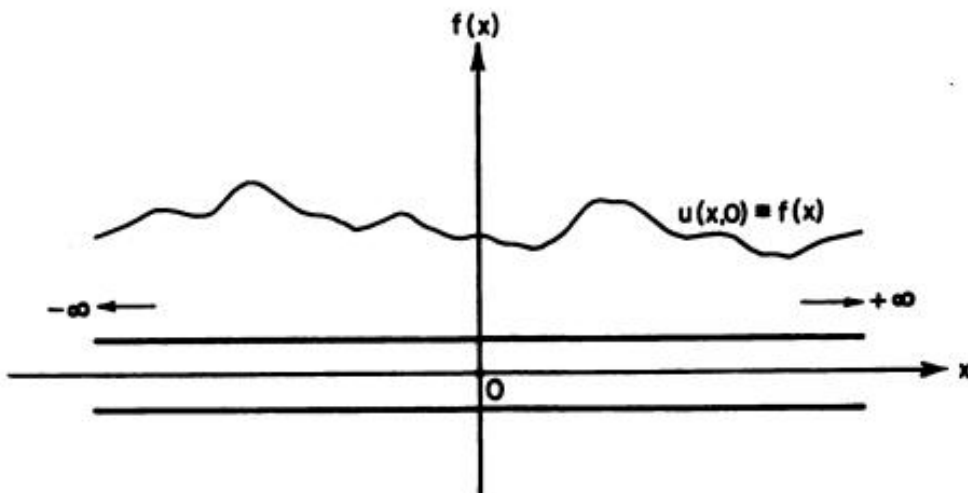


Fig. 57

Assume a solution of Equation (190) of the form

$$u(x, t) = X(x) T(t),$$

which when substituted in Equation (190) yields

$$XT' = c^2 X'' T, \quad \text{or} \quad \frac{T'}{c^2 T} = \frac{X''}{X} = -\beta^2,$$

where β is a real constant. The ordinary differential equations resulting from this equation are

$$X'' + \beta^2 X = 0, \quad T' + c^2 \beta^2 T = 0,$$

and their general solutions are

$$X(x) = A' \cos \beta x + B' \sin \beta x,$$

$$T(t) = C e^{-c^2 \beta^2 t},$$

so that

$$u(x, t) = X(x) T(t) = A e^{-c^2 \beta^2 t} \cos \beta x + B e^{-c^2 \beta^2 t} \sin \beta x, \quad (191)$$

where $A = A' C$, $B = B' C$.

In the case of a bar of finite length, the values of β were determined by one of the boundary conditions at the end of the bar. Here the bar is infinite in length and there are no such boundary conditions. Hence β cannot be determined as was done in the case of a finite bar. Furthermore, since the initial function $f(x) = u(x, 0)$ extends from $-\infty$ to $+\infty$, it is not necessarily periodic and hence may not have a Fourier series expansion.

From the above remarks we see that if this and similar problems are to be solved, the methods of Fourier series must be extended. We shall undertake this extension in the next section in a systematic fashion and thus define the *Fourier integral*. Before undertaking this treatment we can carry the solution of the present problem one step further.

The function u given by Equation (191) satisfies Equation (190) for *any* real value of β . Hence we may write an infinite, continuous set of solutions

$$u_\beta(x, t) = A(\beta) e^{-c^2 \beta^2 t} \cos \beta x + B(\beta) e^{-c^2 \beta^2 t} \sin \beta x, \quad (192)$$

where β may take on any real (not necessarily integral) value, and A and B may be different for each value of β , that is, A and B are functions of β . Since Equation (192) satisfies the linear partial differential equation of Equation (190) for any values of β , the linear combination of an infinity of functions, u_β ,

$$u(x, t) = K \int_0^\infty [A(\beta)e^{-c^2\beta^2t} \cos \beta x + B(\beta)e^{-c^2\beta^2t} \sin \beta x] d\beta, \quad (193)$$

where K is an arbitrary constant, will also satisfy Equation (190).*

59. The sine and cosine transforms

Given a function $f(t)$, not necessarily periodic, extending from $-\infty$ to $+\infty$ (see Figure 58), we wish to obtain for it an expression analogous to the Fourier series expansion, but valid between $-\infty$ and $+\infty$. This can be done as follows :

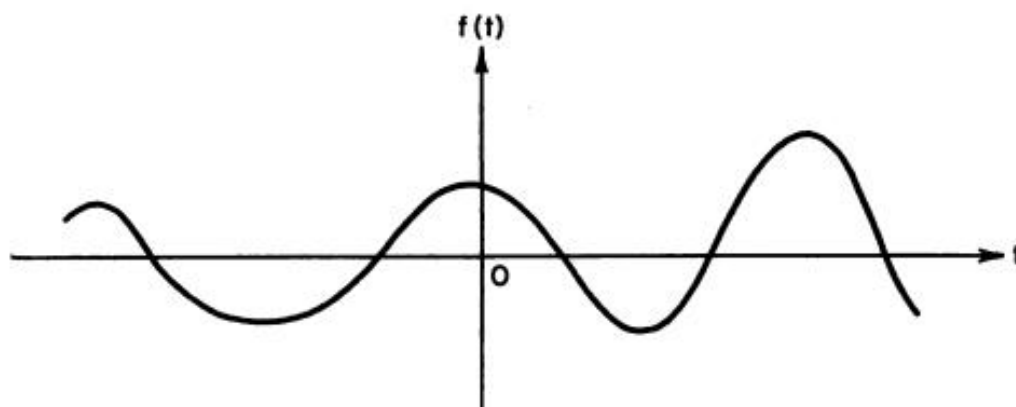


Fig. 58

Define a function $f_T(t)$ such that

$$\begin{aligned} f_T(t) &= f(t) & -\frac{T}{2} < t < \frac{T}{2}, \\ f_T(t) &= 0 & \text{otherwise,} \end{aligned}$$

* There is no necessity for extending the range of integration to $-\infty$, for calling $A'(\beta)$ and $B'(\beta)$ the arbitrary functions $A(-\beta)$ and $-B(-\beta)$, respectively,

$$A(-\beta)e^{-c^2(-\beta)^2t} \cos(-\beta)x = A'(\beta)e^{-c^2\beta^2t} \cos \beta x,$$

and

$$B(-\beta)e^{-c^2(-\beta)^2t} \sin(-\beta)x = B'(\beta)e^{-c^2\beta^2t} \sin \beta x;$$

that is, these terms are of the same form as the terms appearing in the integral of Equation (193).

(see Figure 59). If the graph of f_T is repeated every T units we obtain a periodic function of period T (see Figure 60), and presumably, if T is increased without limit, the Fourier series

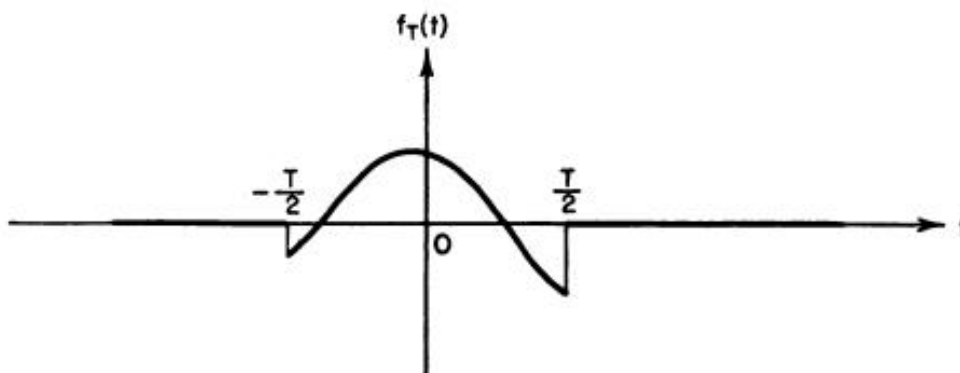


Fig. 59

of $f_T(t)$ will approach that of $f(t)$. Since the function $f_T(t)$ has been prolonged as a periodic function, its Fourier series expansion is given by Equations (90) and (91) where $2L = T$:

$$f_T(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2\pi n}{T} t + b_n \sin \frac{2\pi n}{T} t \right)$$

and

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f_T(t) \cos \frac{2\pi n}{T} t dt,$$

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f_T(t) \sin \frac{2\pi n}{T} t dt.$$

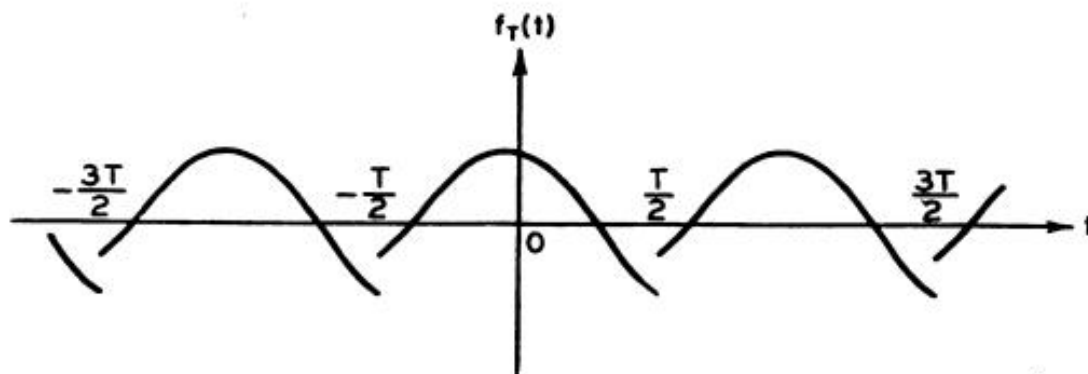


Fig. 60

Letting

$$A_n = \frac{T}{2} a_n, \quad B_n = \frac{T}{2} b_n, \quad \omega_n = \frac{2\pi n}{T},$$

$$f_T(t) = \frac{A_0}{T} + \frac{2}{T} \sum_{n=1}^{\infty} (A_n \cos \omega_n t + B_n \sin \omega_n t), \quad (194)$$

where

$$A_n = \int_{-T/2}^{T/2} f(t) \cos \omega_n t \, dt,$$

$$B_n = \int_{-T/2}^{T/2} f(t) \sin \omega_n t \, dt. \quad (195)$$

Let us now set

$$\Delta\omega_n = \omega_{n+1} - \omega_n = \frac{2\pi(n+1)}{T} - \frac{2\pi n}{T} = \frac{2\pi}{T}$$

in Equation (194), which then becomes

$$f_T(t) = \frac{A_0}{T} + \frac{1}{\pi} \sum_{n=1}^{\infty} A_n \cos \omega_n t \Delta\omega_n + \frac{1}{\pi} \sum_{n=1}^{\infty} B_n \sin \omega_n t \Delta\omega_n.$$

But since $\omega_0 = 0$, and hence $\cos \omega_0 t = 1$, $\sin \omega_0 t = 0$, we may write

$$\frac{A_0}{T} = \frac{A_0 \Delta\omega_0}{2\pi} = \frac{A_0 \cos \omega_0 t}{2\pi} \Delta\omega_0,$$

and adding and subtracting A_0/T to the last equation,

$$f_T(t) = -\frac{A_0}{T} + \frac{1}{\pi} \sum_{n=0}^{\infty} (A_n \cos \omega_n t \Delta\omega_n + B_n \sin \omega_n t \Delta\omega_n). \quad (196)$$

It seems reasonable to surmise that as $T \rightarrow \infty$, the above infinite sums approach integrals. We shall try to make this limiting process appear plausible. The integral

$$\int_0^T A(\omega) \cos \omega t \, d\omega$$

can be closely approximated using a finite sum by the very definition of the definite integral (see [24], Section 1.9). Suppose we divide the ω -axis from 0 to T into N pieces, each of length $2\pi/T$. Clearly, the larger T , the greater N , and the smaller the length of each interval. Let

$$\omega_n = \frac{2\pi n}{T}, \quad n = 0, 1, 2, \dots, N.$$

Then, approximately,

$$\int_0^T A(\omega) \cos \omega t d\omega \doteq \sum_{n=0}^{N-1} A(\omega_n) \cos \omega_n t \Delta\omega_n,$$

where $\Delta\omega_n = \omega_{n+1} - \omega_n = 2\pi/T$. The larger T , the better the approximation.

With this justification, and noting that

$$\lim_{T \rightarrow \infty} \frac{A_0}{T} = 0,$$

Equation (196) assumes the form

$$f(t) = \frac{1}{\pi} \int_0^\infty [A(\omega) \cos \omega t + B(\omega) \sin \omega t] d\omega, \quad (197)$$

and by an even simpler argument

$$\begin{aligned} A(\omega) &= \int_{-\infty}^\infty f(t) \cos \omega t dt, \\ B(\omega) &= \int_{-\infty}^\infty f(t) \sin \omega t dt. \end{aligned} \quad (198)$$

Equation (197) is one form of the so-called Fourier integral. The functions $A(\omega)$ and $B(\omega)$ are called the *cosine* and *sine transforms* or *Fourier transforms* or *spectrum* of the function $f(t)$. Conversely, $f(t)$ is often spoken of as the *inverse Fourier transform* of $A(\omega)$, $B(\omega)$.

Note that while in a *Fourier series* we get contributions only from integral values n of $\omega T/2\pi$, in a *Fourier integral*, we get contributions from *every* real value of ω .

We leave as an exercise for the reader the proof of the following statement analogous to those for Fourier series: If $f(t)$ is an even function, $B(\omega) \equiv 0$, and if $f(t)$ is an odd function, $A(\omega) \equiv 0$.

60. Continuation of the solution of Section 58

We shall use the results of Section 59 to continue the solution of the heat flow problem started in Section 58. Returning to Equation (193), and letting $K = 1/\pi$, $u(x, t)$ becomes

$$u(x, t) = \frac{1}{\pi} \int_0^\infty [A(\beta) e^{-c^2 \beta^2 t} \cos \beta x + B(\beta) e^{-c^2 \beta^2 t} \sin \beta x] d\beta, \quad (199)$$

so that the initial condition $u(x,0) = f(x)$ takes the form

$$u(x,0) = f(x) = \frac{1}{\pi} \int_0^\infty [A(\beta) \cos \beta x + B(\beta) \sin \beta x] d\beta.$$

This equation shows that the unknown functions $A(\beta)$ and $B(\beta)$ are simply the cosine and sine transforms of $f(x)$, and hence, by Equation (198), are given by

$$A(\beta) = \int_{-\infty}^\infty f(x) \cos \beta x dx,$$

$$B(\beta) = \int_{-\infty}^\infty f(x) \sin \beta x dx.$$

By actually substituting the cosine and sine transforms given above into Equation (199),

$$\begin{aligned} u(x,t) &= \frac{1}{\pi} \int_0^\infty \left\{ \left[\int_{-\infty}^\infty f(\zeta) \cos \beta \zeta d\zeta \right] e^{-c^2 \beta^2 t} \cos \beta x \right. \\ &\quad \left. + \left[\int_{-\infty}^\infty f(\zeta) \sin \beta \zeta d\zeta \right] e^{-c^2 \beta^2 t} \sin \beta x \right\} d\beta \\ &= \frac{1}{\pi} \int_{-\infty}^\infty f(\zeta) \left[\int_0^\infty (\cos \beta \zeta \cos \beta x + \sin \beta \zeta \sin \beta x) e^{-c^2 \beta^2 t} d\beta \right] d\zeta \end{aligned}$$

on changing the order of integration. Recalling the trigonometric identity

$$\cos \beta \zeta \cos \beta x + \sin \beta \zeta \sin \beta x = \cos \beta(\zeta - x),$$

$u(x,t)$ assumes the form

$$u(x,t) = \frac{1}{\pi} \int_{-\infty}^\infty f(\zeta) \left[\int_0^\infty e^{-c^2 \beta^2 t} \cos \beta(\zeta - x) d\beta \right] d\zeta. \quad (200)$$

This is the solution of our problem whatever $f(x)$. However, the second integral

$$I = \int_0^\infty e^{-c^2 \beta^2 t} \cos \beta(\zeta - x) d\beta \quad (201)$$

is a definite integral involving only *known* functions, and if it can be evaluated, Equation (200) can be put into a simpler form. We shall compute this integral in the following section.

61. A digression on integration

We shall evaluate certain definite integrals that will be useful in the completion of the solution of the previous section. Consider the integral

$$J = \int_0^{\infty} e^{-a^2 x^2} dx$$

and its square

$$J^2 = \left(\int_0^{\infty} e^{-a^2 x^2} dx \right) \left(\int_0^{\infty} e^{-a^2 y^2} dy \right),$$

which may be written as the double integral

$$J^2 = \int_0^{\infty} \int_0^{\infty} e^{-a^2(x^2+y^2)} dx dy. \quad (202)$$

If we introduce in the above integral a change of variable to polar coordinates,

$$x = r \cos \theta, \quad y = r \sin \theta$$

so that as x and y vary from 0 to ∞ (the first quadrant), r varies from 0 to ∞ , θ varies from 0 to $\pi/2$ and $dx dy = r dr d\theta$, Equation (202) assumes the form

$$J^2 = \int_0^{\pi/2} d\theta \int_0^{\infty} r e^{-a^2 r^2} dr.$$

The value of this integral is

$$J^2 = \frac{\pi}{2} \left(\frac{e^{-a^2 r^2}}{-2a^2} \right)_0^{\infty} = \frac{\pi}{4a^2},$$

and hence

$$J = \int_0^{\infty} e^{-a^2 x^2} dx = \frac{\sqrt{\pi}}{2a}.$$

We can now proceed with the evaluation of the integral

$$I(a, b) = \int_0^{\infty} e^{-a^2 x^2} \cos bx dx,$$

which is a function of the two parameters a and b . If we differentiate I with respect to b and integrate by parts,

$$\frac{dI}{db} = - \int_0^{\infty} x e^{-a^2 x^2} \sin bx dx = \frac{1}{2a^2} \int_0^{\infty} \sin bx d(e^{-a^2 x^2})$$

$$\begin{aligned}
 &= \frac{1}{2a^3} [(\sin bx)e^{-a^2x^2}] \Big|_0^\infty - \frac{1}{2a^3} \int_0^\infty be^{-a^2x^2} \cos bx \, dx \\
 &= 0 - \frac{b}{2a^3} I.
 \end{aligned}$$

Hence

$$\frac{dI}{db} = -\frac{b}{2a^3} I.$$

Integration of this ordinary differential equation yields

$$I(a, b) = Ae^{-b^2/4a^3},$$

where A is a constant of integration. A is evaluated by noting that

$$I(a, 0) = A = \int_0^\infty e^{-a^2x^2} dx = J = \frac{\sqrt{\pi}}{2a}.$$

Hence

$$I(a, b) = \frac{\sqrt{\pi}}{2a} e^{-b^2/4a^3},$$

or

$$\int_0^\infty e^{-a^2x^2} \cos bx \, dx = \frac{\sqrt{\pi}}{2a} e^{-b^2/4a^3}. \quad (203)$$

62. Completion of the solution of Section 58

We are now in a position to complete the solution of the problem started at the beginning of this chapter. Returning to Equation (201), let

$$\beta = x, \quad c^2t = a^3, \quad \zeta - x = b,$$

in Equation (203), which becomes

$$\int_0^\infty e^{-c^2\beta^2t} \cos \beta(\zeta - x) d\beta = \frac{\sqrt{\pi}}{2c\sqrt{t}} \exp \left(-\frac{(\zeta - x)^2}{4c^2t} \right).$$

Substitution of this quantity in Equation (200) gives finally

$$u(x, t) = \frac{1}{2c\sqrt{\pi t}} \int_{-\infty}^\infty f(\zeta) e^{-(\zeta-x)^2/4c^2t} d\zeta,$$

which is the solution of the problem for any $f(x)$.

63. A numerical example

If the function $f(x)$ of Section 58 is defined in the following fashion :

$$\begin{aligned} f(x) &= 0 & x < 0, \\ f(x) &= T \text{ (a constant)} & x > 0, \end{aligned}$$

the solution of the last section becomes

$$u(x,t) = \frac{T}{2c\sqrt{\pi t}} \int_0^\infty e^{-(\zeta-x)^2/4c^2t} d\zeta.$$

If we introduce the change of variable

$$z = (\zeta - x)/2c\sqrt{t} \quad \text{so that} \quad d\zeta = 2c\sqrt{t} dz,$$

and when $\zeta = 0$, $z = -x/2c\sqrt{t}$, and when $\zeta = \infty$, $z = \infty$ then $u(x,t)$ becomes

$$u(x,t) = \frac{T}{\sqrt{\pi}} \int_{-x/2c\sqrt{t}}^\infty e^{-z^2} dz.$$

Now

$$\begin{aligned} \int_{-k}^\infty &= \int_{-k}^0 + \int_0^\infty = \int_0^k + \int_0^\infty, \\ \left(\int_{-k}^0 &= \int_0^k \quad \text{since } e^{-z^2} \text{ is an even function,} \right) \end{aligned}$$

and $u(x,t)$ assumes the form

$$u(x,t) = \frac{T}{\sqrt{\pi}} \int_0^\infty e^{-z^2} dz + \frac{T}{\sqrt{\pi}} \int_0^{x/2c\sqrt{t}} e^{-z^2} dz.$$

Since, as was shown in Section 61,

$$\int_0^\infty e^{-z^2} dz = \frac{\sqrt{\pi}}{2},$$

$$u(x,t) = \frac{T}{2} + \frac{T}{\sqrt{\pi}} \int_0^{x/2c\sqrt{t}} e^{-z^2} dz.$$

The function

$$\frac{2}{\sqrt{\pi}} \int_0^\xi e^{-z^2} dz$$

is known as the *probability integral* or *error function* and is written

$$\operatorname{erf} \xi = \frac{2}{\sqrt{\pi}} \int_0^\xi e^{-z^2} dz.$$

The properties of this function are well known and its values are tabulated, for example, in [22] and [10]. Hence we may write our solution in the compact form

$$u(x,t) = \frac{T}{2} \left(1 + \operatorname{erf} \frac{x}{2c\sqrt{t}} \right). \quad (204)$$

64. Vibrations of an infinite string

As a second example of the use of the Fourier transform we shall determine the displacements $u(x,t)$ of an infinite string. The function $u(x,t)$ satisfies the partial differential equation (Section 2),

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2} \quad (205)$$

and we shall assume the boundary conditions

$$\begin{aligned} \frac{\partial}{\partial t} u(x,0) &= 0 \\ u(x,0) &= f(x) \quad -\infty < x < \infty. \end{aligned} \quad (206)$$

Proceeding as in Section 42, we try a solution of the form $X(x) T(t)$ and find, by using the first boundary condition that u_β ,

$$u_\beta(x,t) = X_\beta(x) T_\beta(t) = [A(\beta) \cos \beta x + B(\beta) \sin \beta x] \cos a\beta t,$$

is a solution of Equation (205) and satisfies the first boundary condition of Equations (206) for *any* real value of β . Hence

$$u(x,t) = \frac{1}{\pi} \int_0^\infty [A(\beta) \cos \beta x + B(\beta) \sin \beta x] \cos a\beta t \, d\beta \quad (207)$$

also has this property. If $u(x,t)$ as given above is to equal $f(x)$ at $t = 0$,

$$u(x,0) = f(x) = \frac{1}{\pi} \int_0^\infty [A(\beta) \cos \beta x + B(\beta) \sin \beta x] \, d\beta, \quad (208)$$

and $A(\beta)$ and $B(\beta)$ must be the cosine and sine transforms of $f(x)$. By Equations (198), therefore,

$$\begin{aligned} A(\beta) &= \int_{-\infty}^\infty f(\zeta) \cos \beta \zeta \, d\zeta, \\ B(\beta) &= \int_{-\infty}^\infty f(\zeta) \sin \beta \zeta \, d\zeta. \end{aligned} \quad (209)$$

Substituting these values for $A(\beta)$ and $B(\beta)$ in Equation (207) and interchanging the order of integration,

$$u(x,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\zeta) \int_0^{\infty} (\cos \beta \zeta \cos \beta x + \sin \beta \zeta \sin \beta x) \cos a\beta t \, d\beta \, d\zeta. \quad (210)$$

By use of the trigonometric identities

$$\cos \beta \zeta \cos \beta x + \sin \beta \zeta \sin \beta x = \cos \beta(\zeta - x)$$

and

$$\cos \beta(\zeta - x) \cos a\beta t = \frac{1}{2} [\cos \beta(\zeta - x + at) + \cos \beta(\zeta - x - at)],$$

Equation (210) becomes

$$u(x,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\zeta) \int_0^{\infty} \frac{1}{2} \cos \beta(\zeta - x + at) \, d\beta \, d\zeta \\ + \frac{1}{\pi} \int_{-\infty}^{\infty} f(\zeta) \int_0^{\infty} \frac{1}{2} \cos \beta(\zeta - x - at) \, d\beta \, d\zeta, \quad (211)$$

which is the desired solution.

We can also determine a form of the solution analogous to that obtained in Section 43. Substituting $A(\beta)$ and $B(\beta)$ as given by Equations (209) into Equation (208), we obtain

$$f(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\zeta) \int_0^{\infty} (\cos \beta \zeta \cos \beta x + \sin \beta \zeta \sin \beta x) \, d\beta \, d\zeta \\ = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\zeta) \int_0^{\infty} \cos \beta(\zeta - x) \, d\beta \, d\zeta.$$

If we replace x by $x \pm at$ in the above equation,

$$f(x \pm at) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\zeta) \int_0^{\infty} \cos \beta(\zeta - x \mp at) \, d\beta \, d\zeta,$$

and comparing this with Equation (211) we see that

$$u(x,t) = \frac{f(x+at) + f(x-at)}{2}.$$

65. The Fourier transform

In this section we shall develop a form of the Fourier integral based on the complex form of the Fourier series. This form is of great importance and has a wide range of applicability even

outside the field of partial differential equations.* Problems solvable by means of the sine and cosine transforms can, of course, be solved by the complex form of the Fourier transform; but this form allows the use of a method totally different from separation of variables for the solution of partial differential equations.

The complex form of the Fourier series of a function $f_T(t)$ (Figure 60) is given (Equation 116) in the notation of this chapter, by

$$F(j\omega_n) = \int_{-T/2}^{T/2} f_T(t) e^{-j\omega_n t} dt, \quad (212)$$

$$f_T(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} F(j\omega_n) e^{j\omega_n t}$$

The discrete set of numbers $F(j\omega_n)$ is often called the (discrete) spectrum of the periodic function $f_T(t)$. If we introduce the notation

$$\omega_n = \frac{2\pi n}{T}, \quad \Delta\omega_n = \omega_{n+1} - \omega_n = \frac{2\pi}{T}$$

as in Section 59, the second of Equations (212) becomes

$$f_T(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} F(j\omega_n) e^{j\omega_n t} \Delta\omega_n,$$

and by the same argument used in Section 59, as $T \rightarrow \infty$, Equations (212) become

$$F(j\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt, \quad (213)$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega.$$

The function $F(j\omega)$ is called the *spectrum* or *Fourier transform*

* For example, linear systems, network theory, amplitude modulation, information theory. In these cases t is usually interpreted as "time" and ω as "angular frequency."

of $f(t)$, and $f(t)$ is called the *inverse Fourier transform* of $F(j\omega)$. $F(j\omega)$ and $f(t)$ are also called *Fourier transform pairs*.*

While mathematical rigor has not been stressed in this book,** it is essential to note that if $F(j\omega)$ as given by the first of Equations (213) is to be meaningful, the integral must exist, and for this to be true,

$$\lim_{t \rightarrow \pm\infty} f(t) = 0.$$

66. Examples of Fourier transforms

In this section we shall calculate the transforms of a few functions of practical interest. These results, as well as others, are listed in Table I. For an extensive table of Fourier transforms, see [4].

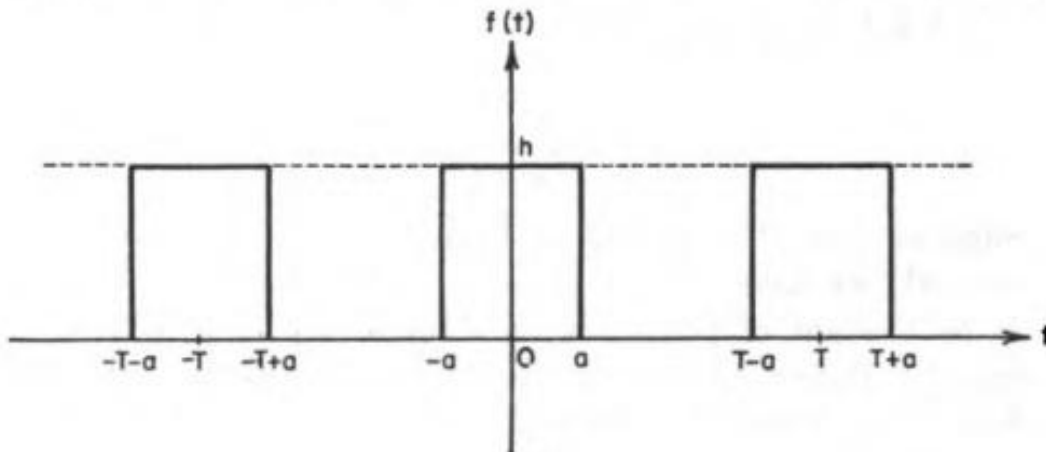


Fig. 61

Consider first a “train of pulses,” that is, the function given in Figure 61 which is periodic of period T , and compute its complex Fourier series by means of Equations (212):

$$\begin{aligned} F(j\omega_n) &= \int_{-T/2}^{T/2} f(t) e^{-j\omega_n t} dt = h \int_{-a}^a e^{-j\omega_n t} dt \\ &= \frac{h}{-j\omega_n} \left(e^{-j\omega_n a} - e^{j\omega_n a} \right) = 2h \frac{\sin \omega_n a}{\omega_n}. \end{aligned}$$

* If $f(t)$ is an *even* function, then $F(j\omega)$ is a *real, even* function of ω . For replacing t by $-t$ in Equation (213),

$$F(j\omega) = \int_{-\infty}^{\infty} f(-t) e^{j\omega t} d(-t) = \int_{-\infty}^{\infty} f(t) e^{j\omega t} dt = F(-j\omega) = \overline{F(j\omega)},$$

and $F(j(-\omega)) = F(-j\omega) = \overline{F(j\omega)} = F(j\omega)$.

** A more detailed discussion of questions involving mathematical rigor appears in Section 70.

We can actually plot $F(j\omega_n)$ (see Figure 62), since this function is real. The ordinates of Figure 62 constitute the discrete spectrum of the function $f(t)$. We also note that the envelope of these discrete ordinates is the continuous function $2h (\sin \omega a)/\omega$.

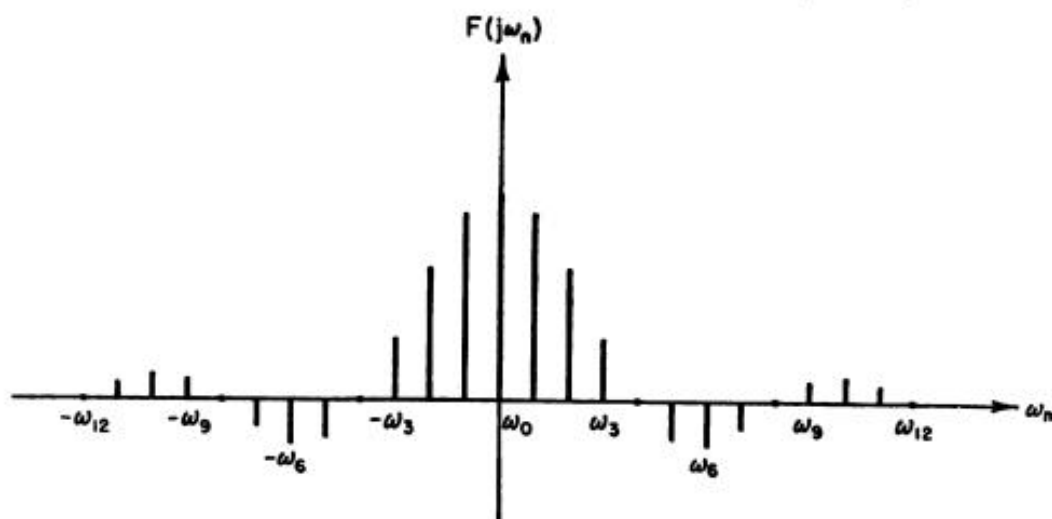


Fig. 62

Suppose now that instead of having a train of pulses, as in Figure 61, we have a *single* pulse as in Figure 63. (This pulse may be thought of physically as a single dot in a Morse code telegraphy system or as a distributed load on an infinite beam resting on an elastic foundation.) This function is not periodic, but if we evaluate its Fourier transform $F(j\omega)$ by means of Equations (213),

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt = h \int_{-a}^a e^{-j\omega t} dt = 2h \frac{\sin \omega a}{\omega},$$

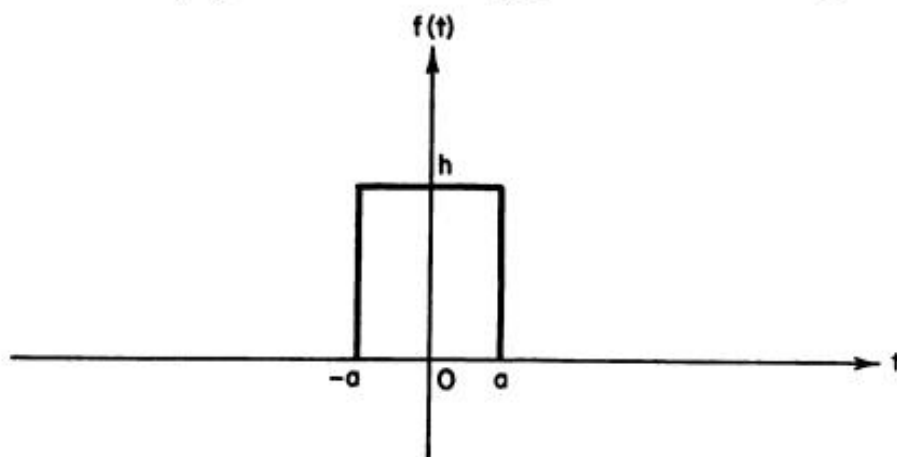


Fig. 63

we see that the Fourier transform of the single pulse is the *envelope* of the lines of the discrete Fourier transform of the train of pulses. The function $F(j\omega)$ is plotted in Figure 64.

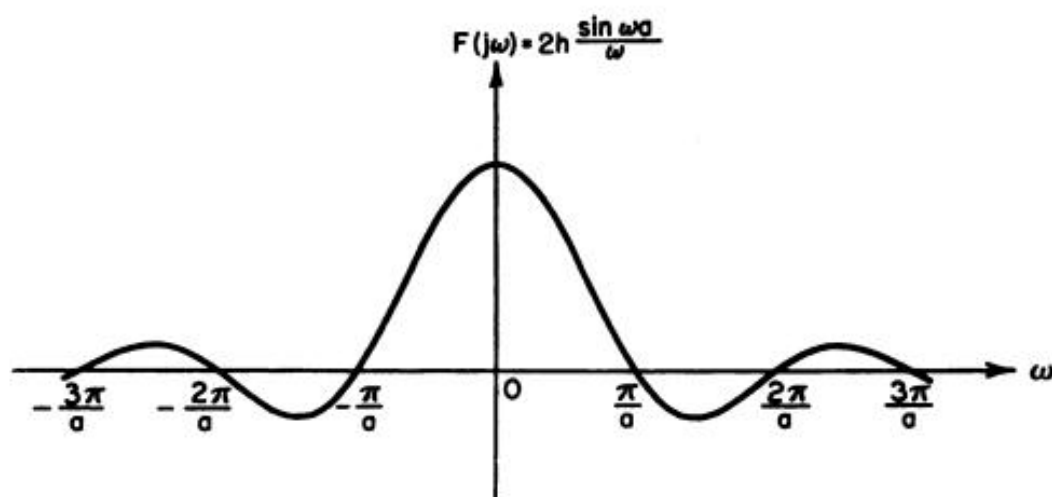


Fig. 64

If we consider the limit of the pulse of Figure 63 obtained by letting $h \rightarrow \infty$ and $a \rightarrow 0$ in such a manner that the area of the pulse, $2ah$, remains constant, say equal to K , we obtain the so called *Dirac delta function*, $\delta(t)$. (In the terminology of the civil engineer $\delta(x)$ represents a concentrated load, and in the terminology of the electrical engineer $\delta(t)$ represents a unit impulse function.) The Fourier transform $\Delta(j\omega)$ of $\delta(t)$ can be obtained as the limit of the Fourier transform of the finite pulse,

$$\Delta(j\omega) = \lim_{\substack{a \rightarrow 0 \\ h \rightarrow \infty}} \frac{2h \sin \omega a}{\omega} = \lim_{a \rightarrow 0} K \frac{\sin \omega a}{\omega a} = K.$$

Hence the Fourier transform of a *unit* impulse function $\delta(t)$ ($K = 1$) is 1.

As the next example we shall calculate the transform of the so-called "unit step function" (Figure 65) which is defined by the equations

$$\begin{aligned} u(t) &= 0 & t < 0, \\ u(t) &= 1 & t > 0. \end{aligned}$$

From Equations (213)

$$U(j\omega) = \mathcal{F}[u(t)] = \int_0^{\infty} e^{-j\omega t} dt = \frac{e^{-j\omega t}}{-j\omega} \Big|_0^{\infty}.*$$

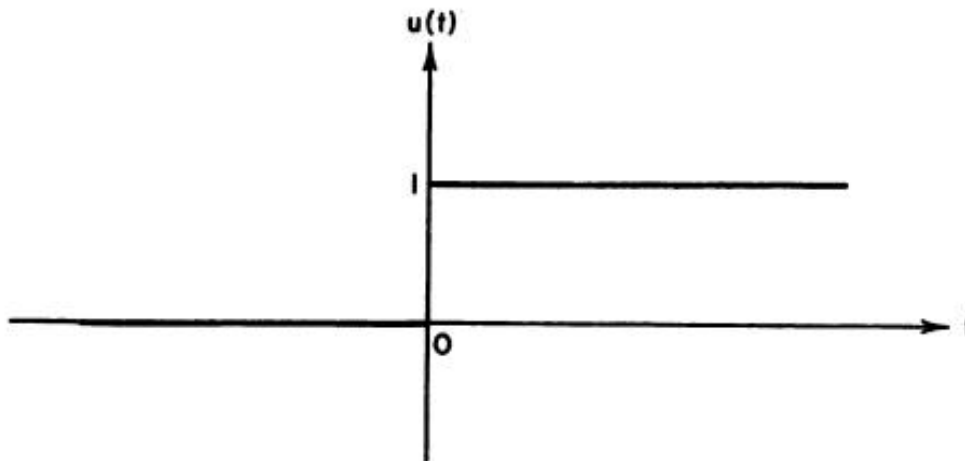


Fig. 65

But

$$\lim_{t \rightarrow \infty} e^{-j\omega t} = \lim_{t \rightarrow \infty} (\cos \omega t - j \sin \omega t)$$

does not exist. We surmount this difficulty by the following device. Consider the function $e^{-\alpha t}$ with $\alpha > 0$. The function $u(t) e^{-\alpha t}$ (Figure 66) is such that

$$\begin{aligned} e^{-\alpha t} u(t) &= 0 & t < 0, \\ e^{-\alpha t} u(t) &= e^{-\alpha t} & t > 0, \end{aligned}$$

and as α approaches zero, $e^{-\alpha t} u(t)$ approaches $u(t)$. The transform of the function $e^{-\alpha t} u(t)$ is easy to calculate by Equations (213):

$$\mathcal{F}[u(t)e^{-\alpha t}] = \int_0^{\infty} e^{-j\omega t} e^{-\alpha t} dt = \frac{1}{\alpha + j\omega}.$$

If we define the transform of $u(t)$ as the limit of the transform of $e^{-\alpha t} u(t)$ as α approaches zero, we find

$$U(j\omega) = \lim_{\alpha \rightarrow 0} \mathcal{F}[e^{-\alpha t} u(t)] = \lim_{\alpha \rightarrow 0} \frac{1}{\alpha + j\omega} = \frac{1}{j\omega}.$$

* We shall use the notation $\mathcal{F}$ to indicate the Fourier transform of a function and write $F(j\omega) = \mathcal{F}[f(t)]$. Correspondingly, we write $f(t) = \mathcal{F}^{-1}[F(j\omega)]$ and read: " $f(t)$ is the inverse Fourier transform of $F(j\omega)$."

Transforms of other elementary functions may be found by using the process above, that is, by introducing a factor $e^{-\alpha t}$ ($\alpha > 0$), computing the transform, and then letting $\alpha \rightarrow 0$.

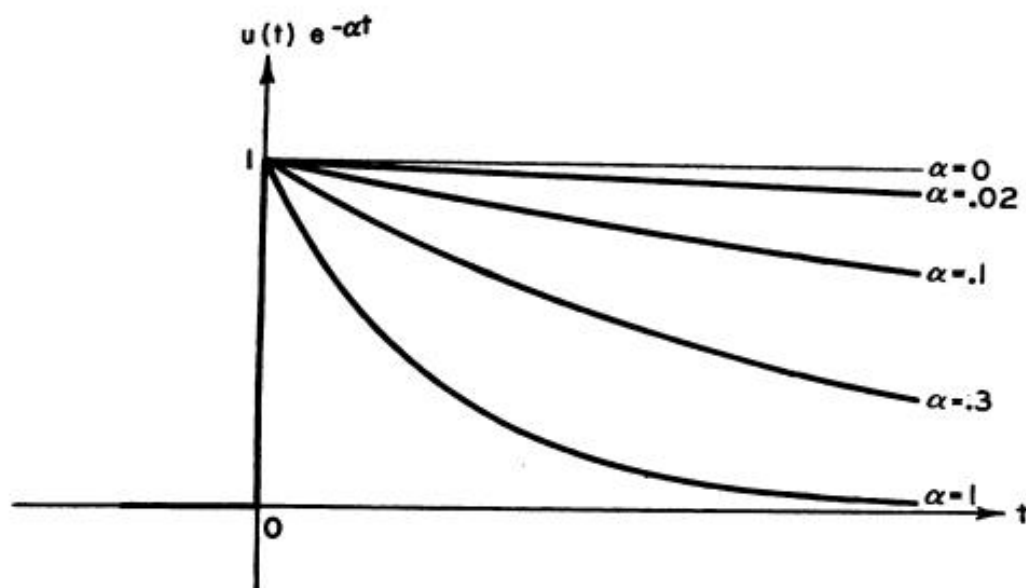


Fig. 66

67. Properties of Fourier transforms

An important property of the Fourier transform is the so-called "translation property" which enables us to calculate the transform of $f(t - \tau)$ (where τ is a constant) in terms of the transform of $f(t)$. By definition

$$\mathcal{F}[f(t - \tau)] = \int_{-\infty}^{\infty} f(t - \tau) e^{-j\omega t} dt$$

where τ is a constant. In the above integral introduce the change of variable: $t = \xi + \tau$. Then

$$\begin{aligned} \mathcal{F}[f(t - \tau)] &= \int_{-\infty}^{\infty} f(\xi) e^{-j\omega(\xi + \tau)} d\xi = e^{-j\omega\tau} \int_{-\infty}^{\infty} f(\xi) e^{-j\omega\xi} d\xi \\ &= e^{-j\omega\tau} F(j\omega), \end{aligned}$$

where $F(j\omega)$ is the Fourier transform of $f(t)$. Hence

$$\mathcal{F}[f(t - \tau)] = e^{-j\omega\tau} \mathcal{F}[f(t)].$$

If we interpret t as "time" and τ as a "time delay," we see that a delay in time is equivalent to multiplication of the transform by an exponential.

TABLE I
FOURIER TRANSFORMS

$f(t)$	$\mathcal{F}[f(t)]$
1. $f(t)$	$F(j\omega)$
2. $f(t - \tau)$	$F(j\omega)e^{-j\omega\tau}$
3. $f'(t)$	$j\omega F(j\omega)$
4. $f^{(n)}(t)$	$(j\omega)^n F(j\omega)$
5. Finite pulse of Fig. 63	$2h \frac{\sin \omega a}{\omega}$
6. $\delta(t)$	1
7. $\delta(t - \tau)$	$e^{-j\omega\tau}$
8. $u(t)$	$\frac{1}{j\omega}$
9. $u(t - \tau)$	$\frac{e^{-j\omega\tau}}{j\omega}$
10. $u(t)e^{-\alpha t}, \quad \alpha > 0$	$\frac{1}{\alpha + j\omega}$
11. $u(t - \tau)e^{-\alpha t}, \quad \alpha > 0$	$\frac{e^{-j\omega\tau}e^{-\alpha\tau}}{\alpha + j\omega}$
12. $e^{-\alpha t }, \quad \alpha > 0$	$\frac{2\alpha}{\alpha^2 + \omega^2}$
13. $u(t)t$	$\frac{1}{(j\omega)^2} \equiv -\frac{1}{\omega^2}$
14. $u(t) \sin at$	$\frac{a}{(j\omega)^2 + a^2} \equiv \frac{a}{a^2 - \omega^2}$
15. $u(t) \cos at$	$\frac{j\omega}{(j\omega)^2 + a^2} \equiv \frac{j\omega}{a^2 - \omega^2}$

Another important transform formula gives the expression for the transform of the *derivative* of a function $f(t)$. Write

$$\mathcal{F}[f'(t)] = \int_{-\infty}^{\infty} f'(t) e^{-j\omega t} dt.$$

If we integrate the above integral by parts,

$$\mathcal{F}[f'(t)] = f(t)e^{-j\omega t} \Big|_{-\infty}^{+\infty} + (j\omega) \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt,$$

and assume, as was remarked in Section 65, that

$$\lim_{t \rightarrow \pm\infty} f(t) = 0,$$

we find that

$$\mathcal{F}[f'(t)] = j\omega F(j\omega),$$

where $F(j\omega)$ is the transform of $f(t)$. We see, then, that differentiation with respect to t is equivalent to multiplication of the transform by $j\omega$. If $f'(t)$ also vanishes at $\pm\infty$, we can compute the Fourier transform of $f''(t)$ in the same fashion,

$$\mathcal{F}[f''(t)] = (j\omega)^2 F(j\omega) = -\omega^2 F(j\omega),$$

and similarly for higher derivatives.

In all the examples given above, transforms were computed starting with a function of time (t). It is clear that we could instead have started with the transform (a function of ω) and obtained the time function by use of the inverse Fourier transform formulas Equation (213). However, in general, this is not an easy process, and frequently necessitates the evaluation of integrals by contour integration in the complex plane—a topic beyond the scope of this book.

68. Transform of the heat equation

In this section we shall apply the Fourier transform to the solution of the heat flow problem solved in Section 62. We recall that the boundary value problem was given by

$$\begin{aligned} \frac{\partial v}{\partial t} &= c^2 \frac{\partial^2 v}{\partial x^2}, \\ v(x, 0) &= f(x) \quad -\infty < x < \infty. \end{aligned} \tag{214}$$

We shall take the transform of the above differential system, solve for the transform of the unknown function v , and apply the inverse Fourier transform of Equation (213). The transform of $v(x, t)$ will be taken with respect to x , letting t be a parameter. Then, by definition, indicating by $\mathcal{F}_x$ the Fourier transform taken with respect to x ,

$$\mathcal{F}_x[v(x, t)] = V(j\xi, t) = \int_{-\infty}^{\infty} v(x, t)e^{-j\xi x} dx$$

(where ξ is the transformed variable corresponding to x). From Table I,

$$\mathcal{F}_x\left[\frac{\partial^2}{\partial x^2} v(x, t)\right] = (j\xi)^2 V(j\xi, t) = -\xi^2 V(j\xi, t), \quad (215)$$

while

$$\begin{aligned} \mathcal{F}_x\left[\frac{\partial}{\partial t} v(x, t)\right] &= \int_{-\infty}^{\infty} \frac{\partial}{\partial t} v(x, t)e^{-j\xi x} dx \\ &= \frac{\partial}{\partial t} \left[\int_{-\infty}^{\infty} v(x, t)e^{-j\xi x} dx \right] = \frac{\partial}{\partial t} V(j\xi, t). \end{aligned} \quad (216)$$

Operating by the Fourier transform on both sides of Equation (214) and using Equations (215) and (216), we obtain the transformed equation

$$\frac{d}{dt} V(j\xi, t) = -c^2 \xi^2 V(j\xi, t). \quad (217)$$

The transformed boundary condition is

$$V(j\xi, 0) = \mathcal{F}_x[v(x, 0)] = \mathcal{F}_x[f(x)] = F(j\xi). \quad (218)$$

Equations (217) and (218) form an *ordinary* differential system on V . The general solution of Equation (217) is

$$V(j\xi, t) = A(j\xi)e^{-c^2 \xi^2 t},$$

where A , the "constant of integration" is a function of ξ . From the boundary condition of Equation (218),

$$F(j\xi) = V(j\xi, 0) = A(j\xi)e^0 = A(j\xi),$$

and hence

$$V(j\xi, t) = F(j\xi)e^{-c^2 \xi^2 t}. \quad (219)$$

The desired solution $v(x, t)$ is the inverse Fourier transform (with respect to x) of $V(j\xi, t)$, and by Equations (213),

$$v(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\xi) e^{-c^2 \xi^2 t} e^{j\xi x} d\xi, \quad (220)$$

which is the required result.

If, for example, as in Section 63 we let $f(x) = Tu(x)$, then

$$F(j\xi) = \mathcal{F}_x[Tu(x)] = \frac{T}{j\xi}$$

and

$$v(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{T}{j\xi} e^{-c^2 \xi^2 t} e^{j\xi x} d\xi. \quad (221)$$

The inverse transform of $(1/j\xi)e^{-c^2 \xi^2 t}$ can be found in [4]. In fact, Formula 727 of [4] yields (in our notation)

$$\mathcal{F}^{-1} \left[\frac{1}{j\xi} e^{-c^2 \xi^2 t} - \frac{1}{j\xi} \right] = \frac{1}{2} \left(\operatorname{erf} \frac{x}{2c\sqrt{t}} - 1 \right),$$

or

$$\mathcal{F}^{-1} \left(\frac{1}{j\xi} e^{-c^2 \xi^2 t} \right) = \frac{1}{2} \left(1 + \operatorname{erf} \frac{x}{2c\sqrt{t}} \right).$$

Hence we conclude as in Section 63 that

$$v(x, t) = \frac{T}{2} \left(1 + \operatorname{erf} \frac{x}{2c\sqrt{t}} \right). \quad (222)$$

69. Transform of the wave equation

As a second example of this technique we shall solve the illustrative problem of Section 64. The boundary value problem is

$$\begin{aligned} \frac{\partial^2 v}{\partial t^2} &= a^2 \frac{\partial^2 v}{\partial x^2}, \\ v(x, 0) &= f(x) \quad -\infty < x < \infty, \\ \frac{\partial}{\partial t} v(x, 0) &= 0. \end{aligned} \quad (223)$$

Transforming this differential system with respect to x we have, letting $V(j\xi, t) = \mathcal{F}_x[v(x, t)]$,

$$\mathcal{F}_x \left[\frac{\partial^2}{\partial x^2} v(x, t) \right] = (j\xi)^2 V(j\xi, t), \quad (224)$$

$$\begin{aligned}\mathcal{F}_x\left[\frac{\partial^2}{\partial t^2} v(x,t)\right] &= \int_{-\infty}^{\infty} \frac{\partial^2}{\partial t^2} v(x,t) e^{-j\xi x} dx \\ &= \frac{\partial^2}{\partial t^2} V(j\xi, t),\end{aligned}\quad (225)$$

$$\mathcal{F}_x[v(x,0)] = V(j\xi,0) = \mathcal{F}[f(x)] = F(j\xi), \quad (226)$$

$$\mathcal{F}_x\left[\frac{\partial}{\partial t} v(x,0)\right] = \frac{\partial}{\partial t} \mathcal{F}_x[v(x,0)] = \frac{\partial}{\partial t} V(j\xi,0). \quad (227)$$

From Equations (224), (225), (226), (227) we obtain the ordinary differential system

$$\frac{d^2}{dt^2} V(j\xi, t) = a^2(j\xi)^2 V(j\xi, t), \quad (228)$$

$$V(j\xi, 0) = F(j\xi), \quad (229)$$

$$\frac{d}{dt} V(j\xi, 0) = 0,$$

on the transformed function V . The general solution of Equation (228) is

$$V(j\xi, t) = A(j\xi)e^{ja\xi t} + B(j\xi)e^{-ja\xi t}.$$

The constants of integration A and B (with respect to t) can be evaluated from Equations (229):

$$F(j\xi) = V(j\xi, 0) = A(j\xi) + B(j\xi),$$

$$0 = \frac{d}{dt} V(j\xi, 0) = ja\xi A(j\xi) - ja\xi B(j\xi).$$

We conclude from these two algebraic equations that

$$A(j\xi) = B(j\xi) = \frac{1}{2} F(j\xi)$$

and hence that

$$V(j\xi, t) = \frac{1}{2} F(j\xi) (e^{ja\xi t} + e^{-ja\xi t}). \quad (230)$$

The desired function $v(x,t)$ is the inverse Fourier transform of $V(j\xi, t)$, namely,

$$\begin{aligned}v(x,t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} V(j\xi, t) e^{j\xi x} d\xi \\ &= \frac{1}{4\pi} \int_{-\infty}^{\infty} F(j\xi) (e^{ja\xi t} + e^{-ja\xi t}) e^{j\xi x} d\xi.\end{aligned}\quad (231)$$

This equation gives the solution of the problem for any $f(x)$. By means of the translation property (see Table I),

$$\begin{aligned}\mathcal{F}_x^{-1}[F(j\xi)e^{ja\xi t}] &= f(x + at), \\ \mathcal{F}_x^{-1}[F(j\xi)e^{-ja\xi t}] &= f(x - at).\end{aligned}$$

hence

$$\begin{aligned}v(x, t) &= \frac{1}{2} \{ \mathcal{F}_x^{-1}[F(j\xi)e^{ja\xi t}] + \mathcal{F}_x^{-1}[F(j\xi)e^{-ja\xi t}] \} \\ &= \frac{1}{2} [f(x + at) + f(x - at)],\end{aligned}\quad (232)$$

which is the familiar equation for the traveling wave.

70. Mathematical conditions on the Fourier integral

The Fourier transform is defined *only* for functions which are *absolutely integrable*, that is for functions $f(x)$ such that

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty. \quad (233)$$

The Fourier transform $F(\omega)$ of such an $f(x)$ is defined as

$$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-j\omega x} dx, \quad (234)$$

and can be shown to be a continuous function of ω .

On the other hand, Equation (234) does *not* imply that $f(x)$ is the inverse Fourier transform of $F(\omega)$, that is, that

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega x} d\omega, \quad (235)$$

unless certain additional conditions are satisfied. A *sufficient* condition for Equation (235) to be valid at a point $x = x_0$ is that the right- and left-hand derivatives of $f(x)$ exist at $x = x_0$,* in which case

$$\frac{1}{2} [f(x_0+) + f(x_0-)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega x_0} d\omega. \quad (236)$$

* By definition, the left-hand derivative of $f(x)$ at the point $x = x_0$ is

$$f'(x_0-) = \lim_{\substack{x \rightarrow x_0 \\ x < x_0}} f'(x),$$

and the right-hand derivative of $f(x)$ at the point $x = x_0$ is

$$f'(x_0+) = \lim_{\substack{x \rightarrow x_0 \\ x > x_0}} f'(x)$$

(see Section 26).

If, moreover, $f(x)$ is continuous at $x = x_0$, then $f(x_0+) = f(x_0-)$, and Equation (236) becomes identical with Equation (235).

Equation (235) is valid for *all* x (and not just at a point as described above) under various sets of hypotheses. Perhaps the simplest of these is given by the following theorem.

THEOREM. If $f(x)$ and $F(\omega)$ are absolutely integrable, $f(x)$ is continuous, and

$$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-j\omega x} dx,$$

then

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega x} d\omega$$

holds for *all* x .

For a further discussion of the subject including a wide variety of theorems and proofs, see [2].

PROBLEMS

1. Show that if $f(t)$ is an even function (see Section 59), $B(\omega) \equiv 0$, and if $f(t)$ is an odd function, $A(\omega) \equiv 0$.

2. Find the cosine and sine Fourier transforms $A(\omega)$ and $B(\omega)$ of the function $f(\xi)$ given in Figure 67.

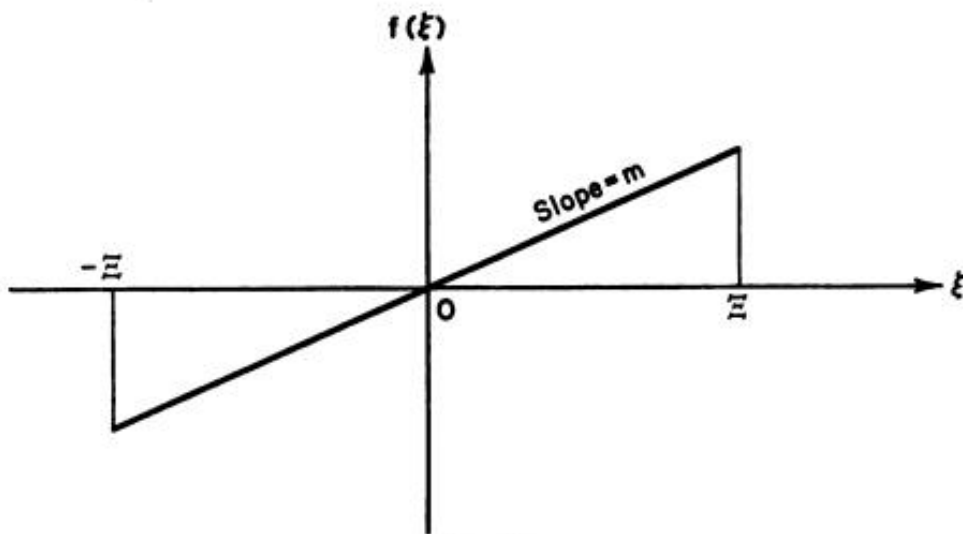


Fig. 67

3. Find $A(\omega)$ and $B(\omega)$ for the function of Figure 68.

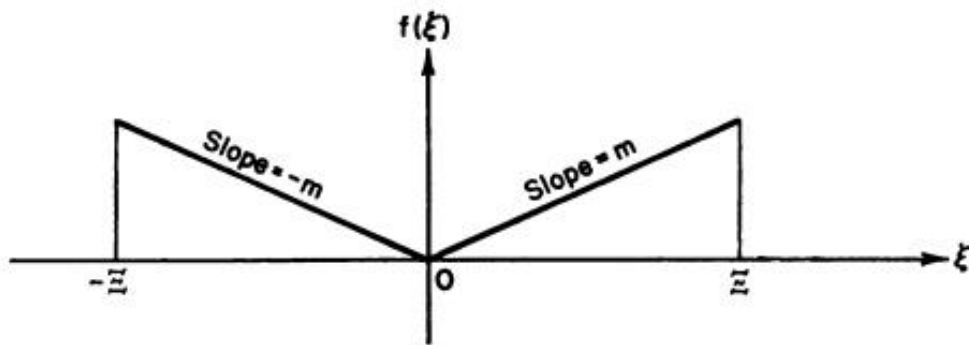


Fig. 68

4. Find $A(\omega)$ and $B(\omega)$ for the function of Figure 69.

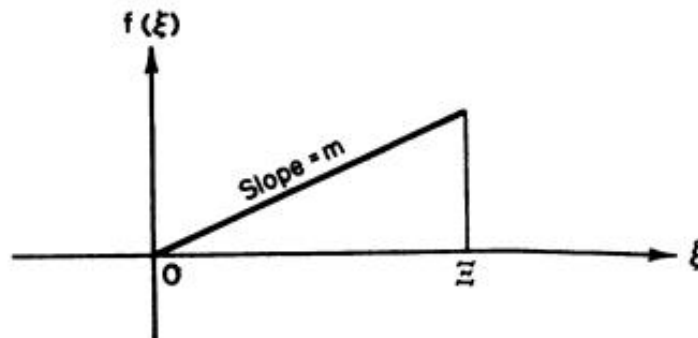


Fig. 69

5. Find $A(\omega)$ and $B(\omega)$ for the function of Figure 70.

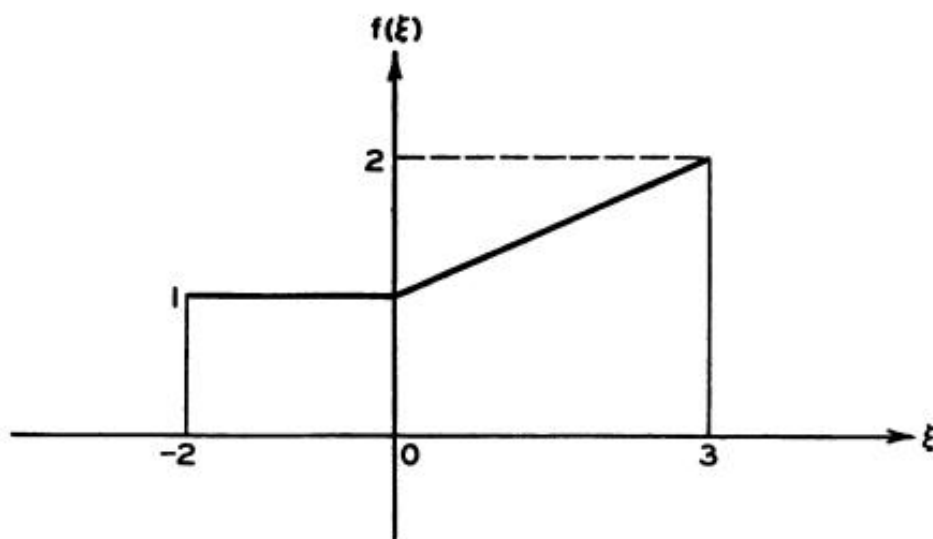


Fig. 70

6. Find $A(\omega)$ and $B(\omega)$ for the function $f(\xi)$:

$$\begin{aligned} f(\xi) &= 1 - \xi^2 & -1 < \xi < 1, \\ f(\xi) &= 0 & \text{otherwise.} \end{aligned}$$

7. Plot the functions $A(\omega)$ and $B(\omega)$ for Problems 2, 3, and 4 so that they can be used for any values of the parameters Ξ and m , that is, use nondimensional parameters.

8. Find the Fourier transform $F(j\omega)$ of each of the functions of Problems 2, 3, 4, 5, and 6, and plot the magnitude and phase of $F(j\omega)$ for each.

9. The lateral surface of a semi-infinite bar (one-dimensional case) is insulated, and the end $x = 0$ is kept at zero temperature. Find the temperature distribution $u(x, t)$ if the initial temperature $u(x, 0)$ is a constant T . Plot this temperature distribution between $x = 0$ and $x = 10$ at $t = 4$ sec and $t = 10$ sec. Assume $k/c\delta = 0.25$, where k = thermal conductivity, c = specific heat, δ = density.

10. Why is it necessary to assume $u(x, 0) = -T$ for negative values of x in Problem 9?

11. Find the temperature $u(x, t)$ in an infinite bar in which the initial temperature distribution is

$$u(x, 0) = Te^{-\alpha|x|}, \quad \alpha > 0.$$

Assume the lateral surface of the bar is insulated.

12. The lateral surface of a semi-infinite bar is insulated and at its end $x = 0$ the temperature gradient

$$\frac{\partial}{\partial x} u(0, t)$$

is zero. Find the temperature distribution $u(x, t)$ if $u(x, 0) = f(x)$, $0 < x < \infty$.

13. Consider a thin slab of infinite extent whose lateral surfaces are insulated. Let $\phi(x, y)$ be the initial temperature distribution in the plate. Find the temperature $u(x, y, t)$.

14. Find the temperature distribution in a semi-infinite slab (Figure 71) if the initial temperature distribution is $u(x, y, 0) = f(x, y)$, $0 < x < \infty$, $-\infty < y < \infty$, and edge of the plate, $x = 0$ is kept at zero temperature. Assume that the lateral surfaces are insulated.

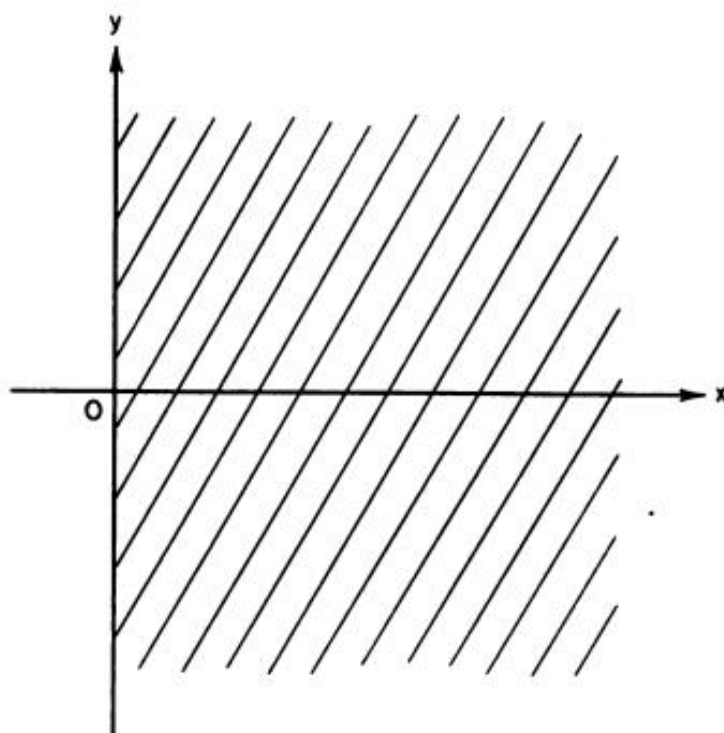


Fig. 71

15. Solve Problem 14 if, instead of having $u(0, y, t) = 0$, this condition is replaced by the adiabatic condition

$$\frac{\partial}{\partial x} u(0, y, t) = 0.$$

16. Consider a "quarter-infinite" plate (Figure 72) insulated on both lateral surfaces. Let the initial temperature distribution be $u(x, y, 0) = \phi(x, y)$, $0 \leq x < \infty$, $0 < y < \infty$. Find $u(x, y, t)$ if the edges $x = 0$ and $y = 0$ are kept at zero temperature.

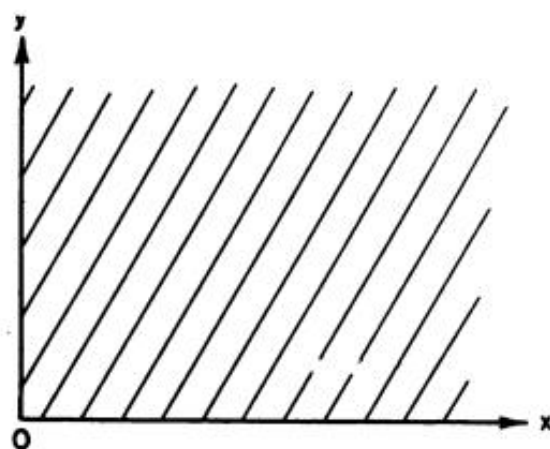


Fig. 72

17. Solve Problem 16 if the conditions $u(0, y, t) = 0 = u(x, 0, t)$ are replaced by adiabatic conditions.

18. Solve Problem 16 if the boundary conditions are

$$1. \quad u(x, y, 0) = \phi(x, y),$$

$$2. \quad u(0, y, t) = 0,$$

$$3. \quad \frac{\partial}{\partial y} u(x, 0, t) = 0.$$

19. A three-dimensional infinite solid has an initial temperature distribution $u(x, y, z, 0) = \phi(x, y, z)$, $-\infty < x < \infty$, $-\infty < y < \infty$, $-\infty < z < \infty$. Find $u(x, y, z, t)$.

20. A semi-infinite string, fixed at the origin (Figure 73) has an initial displacement and velocity given by $f(x)$ and $g(x)$, respectively. Find the transverse displacement $u(x, t)$ of the string.



Fig. 73

21. Solve Problem 20 if the boundary conditions are $u(x, 0) = f(x)$ and

$$\frac{\partial}{\partial t} u(x, 0) = 0.$$

22. Solve Problem 20 with the boundary conditions

$$1. \quad u(0, t) = 0,$$

$$2. \quad u(x, 0) = 0,$$

$$3. \quad \frac{\partial}{\partial t} u(x, 0) = g(x).$$

23. Consider a membrane, infinite in extent, whose initial displacement is given by $u(x, y, 0) = \phi(x, y)$ and whose initial velocity is zero. Find the transverse displacements $u(x, y, t)$ of this membrane.

24. Solve Problem 23 for the boundary conditions

$$1. \quad u(x, y, 0) = 0,$$

$$2. \quad \frac{\partial}{\partial t} u(x, y, 0) = \psi(x, y).$$

25. A semi-infinite membrane (Figure 71) is held fast along the y -axis. Determine the displacement $u(x, y, t)$ if the initial velocity is zero and the initial displacement is $u(x, y, 0) = \phi(x, y)$, $0 < x < \infty$, $-\infty < y < \infty$.

26. Solve Problem 25 if the initial displacement is zero and the initial velocity

$$\frac{\partial}{\partial t} u(x, y, 0)$$

is $\psi(x, y)$.

27. Figure 72 represents a "quarter-infinite" membrane held fixed along the x - and y -axes. Find the displacement $u(x, y, t)$ if $u(x, y, 0) = \phi(x, y)$ and

$$\frac{\partial}{\partial t} u(x, y, 0) = 0.$$

28. Find the displacement $u(x, y, t)$ of a quarter-infinite vibrating membrane if $\phi(x, y) = 0$ and

$$\frac{\partial}{\partial t} u(x, y, 0) = \psi(x, y).$$

29. A semi-infinite shaft, fixed at the origin is vibrating torsionally. Find the equation of motion for $\theta(x, t)$ if the initial angle of twist $\theta(x, 0) = F(x)$ and initial velocity is zero (see Problem 11 of Chapter III).

30. An infinite shaft vibrating torsionally has an initial velocity

$$\frac{\partial}{\partial t} \theta(x, 0) = G(x)$$

and zero initial angle of twist. Find the angle of twist $\theta(x, t)$ at any time t .

31. A high-frequency transmission line, infinite in extent, has an initial voltage distribution $e(x, 0) = E(x)$ and an initial current distribution $i(x, 0) = I(x)$. Find the currents and voltages in the line at any point x and at any time t .

32. The high-frequency line of Problem 31 is semi-infinite in extent, with zero voltage at the sending end (origin). Find $e(x, t)$ and $i(x, t)$ in a line subject to the same boundary conditions as in Problem 31.

33. A submarine cable, semi-infinite in extent has zero voltage at the sending end and an initial voltage distribution $e(x, 0) = E(x)$, $0 < x < \infty$. Find the voltage $e(x, t)$ at any time t .

34. A submarine cable, infinite in extent, has an initial current distribution $i(x, 0) = I(x)$. Find $i(x, t)$.

35. The function

$$\text{Si}(x) = \int_0^x \frac{\sin z}{z} dz$$

is known as the "sine-integral function." Show that $\lim_{x \rightarrow \infty} \text{Si}(x) = \pi/2$.
[Hint: Consider the integral

$$\int \frac{e^{-\alpha z} \sin z}{z} dz$$

where α is a parameter.]

36. Prove :

$$\int_0^1 \frac{x^\alpha - 1}{\log x} dx = \log(1 + \alpha), \alpha \geq 0.$$

37. Show that

$$\int_0^\infty e^{-x^2 - a^2/x^2} dx = \frac{\sqrt{\pi}}{2} e^{-2a}.$$

38. Prove that

$$\int_0^\pi \log(1 - 2t \cos x + t^2) dx = \begin{cases} \pi \log t^2 & \text{if } t^2 \geq 1, \\ 0 & \text{if } t^2 \leq 1. \end{cases}$$

39. Establish the formula

$$\int_0^\infty \frac{e^{-x} \sin x}{x} dx = \frac{\pi}{4}.$$

40. Prove that

$$\mathcal{F}[\delta(t - \tau)] = e^{-j\omega\tau}$$

by considering $\delta(t - \tau)$ as the limiting case of the finite pulse of Figure 74.

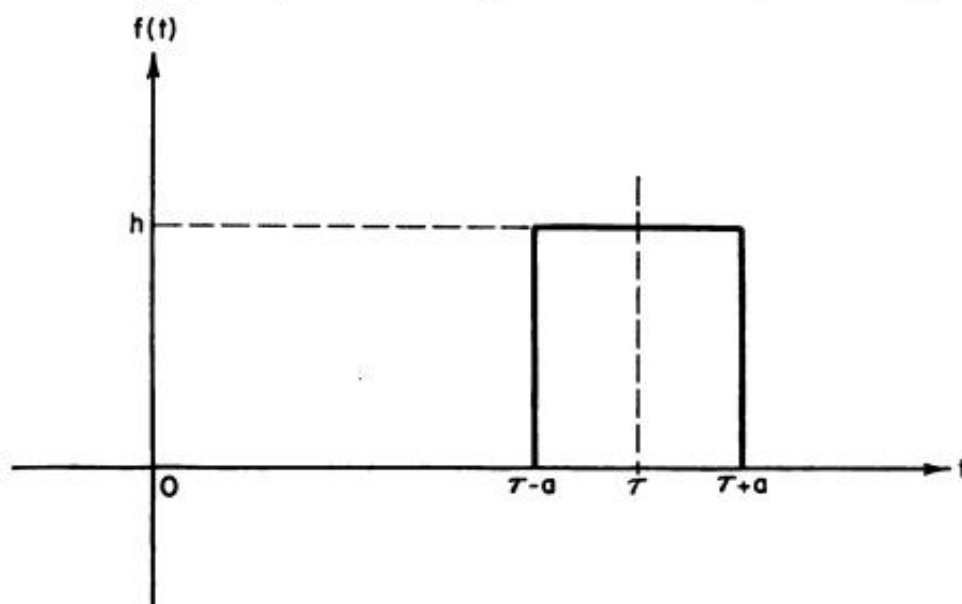


Fig. 74

41. Prove that

$$\mathcal{F}[tu(t)] = \frac{1}{(j\omega)^2},$$

where $u(t)$ is the unit step function.

42. Find the Fourier transforms of the functions

$$u(t) \sin at, \quad u(t) \cos bt,$$

where $u(t)$ is the unit step function.

43. Show that the Fourier transform of $e^{-\alpha|t|}$, $\alpha > 0$ is given by

$$\frac{2\alpha}{\alpha^2 + \omega^2}.$$

44. Calculate the discrete or continuous spectrum of the following functions and plot the results.

(a) The periodic function $\sin t$.

(b) The periodic function $f(t)$ of period $4N\pi$ defined in the interval $(0, 4N\pi)$ as

$$\begin{aligned} f(t) &= \sin t & 0 < t < 2N\pi, \\ f(t) &= 0 & 2N\pi < t < 4N\pi. \end{aligned}$$

(c) The nonperiodic function $g(t)$ defined as follows :

$$\begin{aligned} g(t) &= \sin t & 0 < t < \infty, \\ g(t) &= 0 & -\infty < t < 0. \end{aligned}$$

(d) The nonperiodic function $h(t)$ defined as follows :

$$\begin{aligned} h(t) &= \sin t & -2N\pi < t < \infty, \\ h(t) &= 0 & -\infty < t < -2N\pi. \end{aligned}$$

In parts (b) and (d) assume that N is a large positive integer. Which results can be considered as limiting cases of the others? Discuss these results.

45. By a direct application of the inverse Fourier transform formula [Equations (213)] calculate the inverse Fourier transform of $F(j\omega) = \left(\frac{\sin \omega}{\omega}\right)$.

Solve the following problems by the methods of Sections 68, 69; that is, calculate the Fourier transform of the unknown function and solve the resulting transformed equation for the unknown Fourier transform.

- 46. Problem 9.
- 47. Problem 11.
- 48. Problem 12.
- 49. Problem 20.
- 50. Problem 21.
- 51. Problem 22.
- 52. Problem 29.
- 53. Problem 30.
- 54. Problem 31.
- 55. Problem 32.
- 56. Problem 33.
- 57. Problem 34.

Chapter V

LEGENDRE, BESSEL, AND MATHIEU FUNCTIONS

71. Some important ordinary differential equations

In previous examples when we used the method of separation of variables to solve a partial differential equation, the resulting ordinary differential equations were generally linear with constant coefficients. In this chapter we shall consider certain partial differential equations of engineering importance which give rise to variable coefficient ordinary differential equations of great interest, namely, Legendre's equations, Bessel's equation, and Mathieu's equation.

We shall discuss first certain essential preliminaries such as orthogonal functions and the gamma function, and then solve Legendre's, Bessel's, and Mathieu's equations and study some of their more elementary properties. At this stage of our development we shall be in a position to solve completely the original boundary value problems which led to these ordinary differential equations.

72. Legendre's equation

Consider a sphere of radius A with a certain potential (gravitational, electric, or magnetic) distribution on the surface; we wish to find the potential distribution at interior points of the sphere. The potential $u(x, y, z)$ satisfies the three-dimensional Laplacian

$$\nabla^2 u = 0,$$

where, in Cartesian coordinates,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

However, since our problem concerns a *sphere* it is convenient to transform ∇^2 into *spherical coordinates*, just as in Section 50 where we changed the two-dimensional Laplacian ∇^2 into *polar coordinates* because our problem involved a semicircular slab. The spherical coordinate system we shall use is illustrated in Figure 75

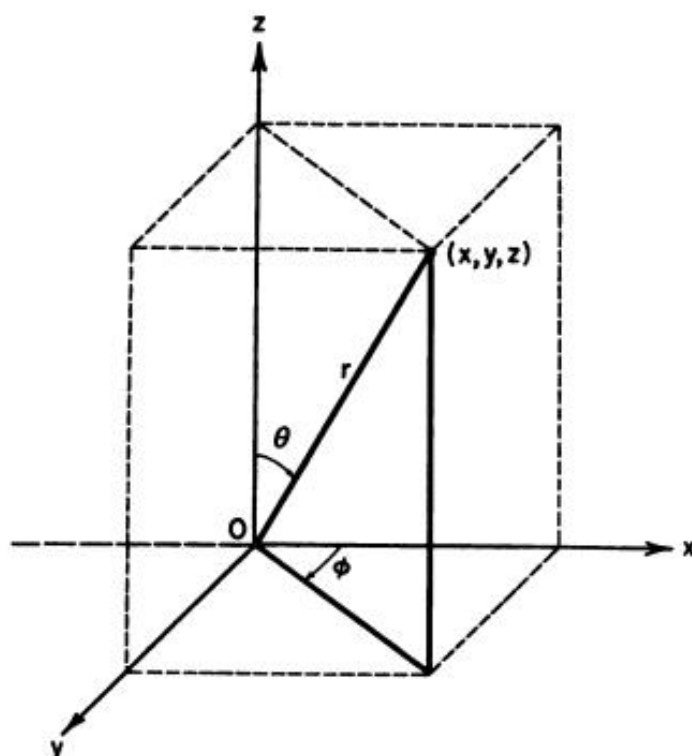


Fig. 75

where in terms of r , θ , and ϕ , the Cartesian coordinates x , y , and z are given by

$$x = r \cos \phi \sin \theta, \quad y = r \sin \phi \sin \theta, \quad z = r \cos \theta.$$

From these equations we may calculate

$$\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \text{ etc.}$$

in terms of r , θ , and ϕ , and write

$$\nabla^2 u = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2}.$$

The details of the partial differentiations follow the same pattern as in Section 50 and will be left as an exercise for the reader.

In particular, if the potential is symmetric about the z -axis,

$$\frac{\partial u}{\partial \phi} = 0,$$

and the above equation assumes the form

$$\sin \theta \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) = 0. \quad (237)$$

The boundary condition, giving the initial potential on the surface of the sphere can now be written

$$u(A, \theta) = f(\theta), \quad (238)$$

where $f(\theta)$ is the given potential distribution.

Equations (237) and (238) form a differential system which we shall attempt to solve by the method of separation of variables. Substituting a solution of the form

$$u(r, \theta) = R(r)\Theta(\theta)$$

into Equation (237), we obtain

$$\Theta \sin \theta \frac{d}{dr} (r^2 R') + R \frac{d}{d\theta} (\Theta' \sin \theta) = 0,$$

or dividing by $R\Theta \sin \theta$,

$$\frac{1}{\Theta \sin \theta} \frac{d}{d\theta} (\Theta' \sin \theta) = - \frac{1}{R} \frac{d}{dr} (r^2 R').$$

By the usual argument, the two sides of this equation must be equal to a constant, say λ , so that

$$\frac{1}{\sin \theta} \frac{d}{d\theta} (\Theta' \sin \theta) - \lambda \Theta = 0, \quad (239a)$$

$$\frac{d}{dr} (r^2 R') + \lambda R = 0. \quad (239b)$$

The second of these equations becomes, on performing the indicated differentiations,

$$r^2 R'' + 2rR' + \lambda R = 0$$

which we recognize as Cauchy's equation (see Section 50). If we introduce the change of variable

$$x = \cos \theta,$$

$$\frac{d}{d\theta} = \frac{dx}{d\theta} \frac{d}{dx} = -\sin \theta \frac{d}{dx}$$

in Equation (239a), this equation assumes the form

$$+ \frac{d}{dx} \left(\sin^2 \theta \frac{d\Theta}{dx} \right) - \lambda \Theta = 0,$$

or

$$\frac{d}{dx} \left[(1 - x^2) \frac{d\Theta}{dx} \right] - \lambda \Theta = 0.$$

This is one of the fundamental equations of mathematical physics and is called *Legendre's equation*. If, as is customary, we let

$$\lambda = -n(n+1),$$

Legendre's equation assumes the classical form

$$\frac{d}{dx} \left[(1 - x^2) \frac{d\Theta}{dx} \right] + n(n+1)\Theta = 0. \quad (240)$$

The properties of this equation are studied in Sections 77-79.

73. Bessel's equation

Suppose we have a thin circular disk insulated on its lateral surfaces, whose temperature depends only on the distance from the center (and not on the direction). We wish to determine the transient temperature distribution in the disk for a given set of boundary conditions. The partial differential equation governing this phenomenon is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{1}{k} \frac{\partial u}{\partial t} = 0.* \quad (241)$$

If we call a the radius of the disk, typical boundary conditions that could be associated with Equation (241) are

$$u(a, t) = 0, \quad u(r, 0) = f(r).$$

* This equation was derived in Problem 18 of Chapter I directly from physical considerations.

These indicate that the temperature on the boundary of the disk is permanently zero, while the initial temperature distribution is a given function of r , say $f(r)$.

As in previous examples, we shall substitute in Equation (241) a solution of the form

$$u(r, t) = R(r) T(t)$$

and obtain by separating the variables,

$$\frac{R''}{R} + \frac{1}{r} \frac{R'}{R} = \frac{1}{k} \frac{T'}{T} = \lambda,$$

where λ is a constant. If λ is negative, $\lambda = -\beta^2$, the solution of the equation in $T(t)$ will contain a negative exponential, as one expects from physical considerations, and the above equation gives rise to the two equations,

$$T' + k\beta^2 T = 0, \quad (242a)$$

$$R'' + \frac{1}{r} R' + \beta^2 R = 0. \quad (242b)$$

The second of these equations is another fundamental equation of mathematical physics and is known as *Bessel's equation*. If we let $s = \beta r$, Equation (242b) becomes

$$\frac{d^2 R}{ds^2} + \frac{1}{s} \frac{dR}{ds} + R = 0,$$

or

$$s^2 \frac{d^2 R}{ds^2} + s \frac{dR}{ds} + s^2 R = 0. \quad (243)$$

This is the classical form of the so-called *Bessel's equation of the zero order*. The more general equation

$$s^2 \frac{d^2 R}{ds^2} + s \frac{dR}{ds} + (s^2 - n^2) R = 0, \quad (244)$$

which arises in other boundary value problems, is known as *Bessel's equation of the n th order*. Clearly Equation (243) is the particular case of Equation (244) corresponding to $n = 0$.

We shall solve this equation and make a study of its properties in later sections (Sections 81-83).

74. Mathieu's equation

Let us consider an *elliptic membrane*, fixed on the boundary and subjected to an initial displacement and an initial velocity. We wish to determine the transverse displacements of the membrane. The partial differential equation (in Cartesian coordinates) governing this phenomenon is (see Section 4),

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2}. \quad (245)$$

It will be convenient at this point to eliminate the time variable. Since essentially we shall assume a solution with the separated variables, and since the free vibrations of the membrane vary sinusoidally with time, we let

$$u(x, y, t) = w(x, y) (A \cos \omega t + B \sin \omega t).$$

Substituting this equation into Equation (245), we obtain

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\omega^2}{a^2} w = 0, \quad (246)$$

which is an equation in the two variables x and y .

Because of the shape of the membrane it is convenient to transform Equation (246) into *elliptic coordinates* (ξ, η) , by the transformation

$$x = \gamma \cosh \xi \cos \eta, \quad y = \gamma \sinh \xi \sin \eta, \quad (247)$$

where γ is a constant. The variable η varies from 0 to 2π , and the variable ξ from 0 to ∞ .

We shall discuss this transformation in some detail. If we let ξ be a constant, Equation (247) defines a locus which is an *ellipse*, since

$$\frac{x}{\gamma \cosh \xi} = \cos \eta, \quad \frac{y}{\gamma \sinh \xi} = \sin \eta,$$

and

$$\frac{x^2}{(\gamma \cosh \xi)^2} + \frac{y^2}{(\gamma \sinh \xi)^2} = 1. \quad (248)$$

The semimajor axis of this ellipse is $\gamma \cosh \xi$, and the semiminor axis is $\gamma \sinh \xi$ (Figure 76). The distance between the foci is

$$2\sqrt{(\gamma \cosh \xi)^2 - (\gamma \sinh \xi)^2} = 2\gamma.$$

Hence we see that the curves $\xi = \text{constant}$ form a family of confocal ellipses. When $\xi = 0$ we obtain the degenerate ellipse which is the straight line between $(-\gamma, 0)$ and $(\gamma, 0)$ (Figure 76). As $\xi \rightarrow \infty$, the ellipse approaches a circle of infinite radius with center at the origin.

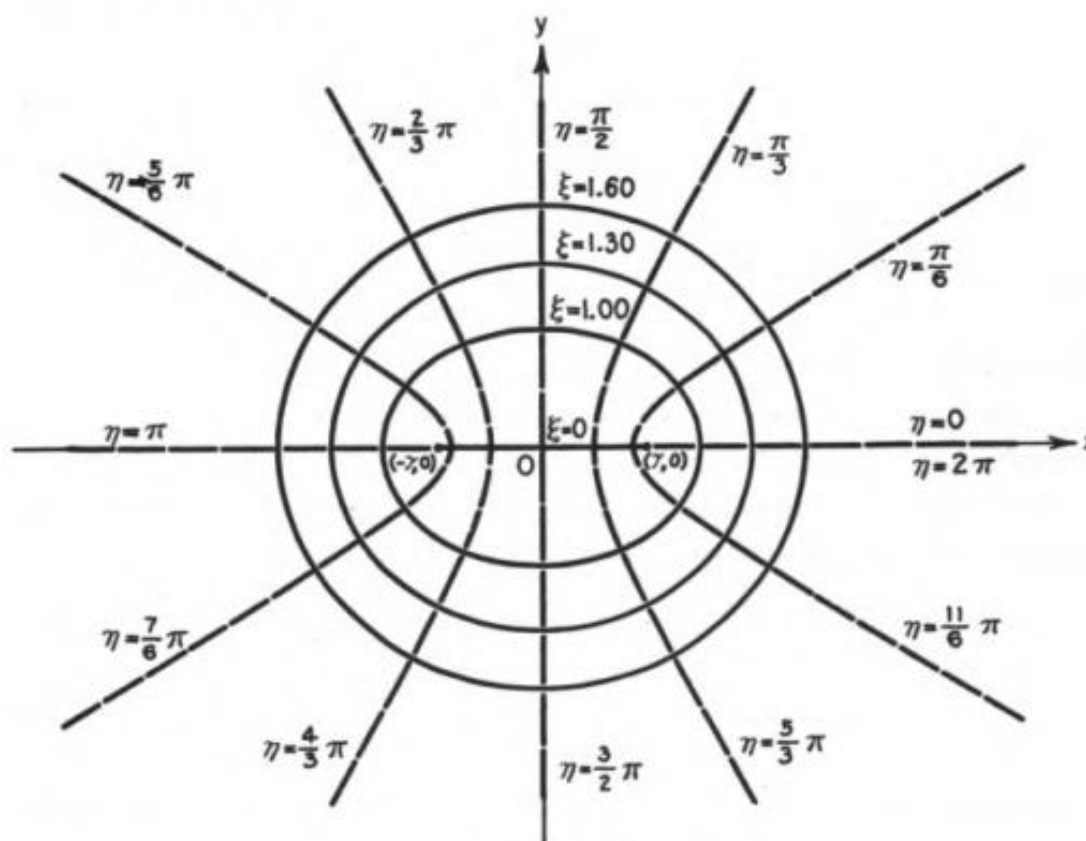


Fig. 76

If, on the other hand, $\eta = \text{constant}$, Equations (247) yield

$$\frac{x}{\gamma \cos \eta} = \cosh \xi, \quad \frac{y}{\gamma \sin \eta} = \sinh \xi,$$

and

$$\frac{x^2}{(\gamma \cos \eta)^2} - \frac{y^2}{(\gamma \sin \eta)^2} = \cosh^2 \xi - \sinh^2 \xi = 1.$$

These curves represent hyperbolas with principal axes coincident with the x -axis. The distance between their foci is

$$2\sqrt{(\gamma \cos \eta)^2 + (\gamma \sin \eta)^2} = 2\gamma.$$

Hence the hyperbolas are confocal and, moreover, have the same foci as the ellipses, $\xi = \text{constant}$.

If $\eta = 0$, we find from Equations (247) that

$$x = \gamma \cosh \xi, \quad y = 0,$$

and the corresponding hyperbola reduces to the straight line extending along the x -axis from $x = \gamma$ to $x = +\infty$. Similarly, if $\eta = \pi$,

$$x = -\gamma \cosh \xi, \quad y = 0,$$

and the hyperbola reduces to the straight line extending along the x -axis from $x = -\gamma$ to $x = -\infty$. Also, if $\eta = \pi/2$,

$$x = 0, \quad y = \gamma \sinh \xi,$$

and the corresponding locus is the upper half of the y -axis. For $\eta = 3\pi/2$ the locus is $x = 0$, $y = -\gamma \sinh \xi$, or that portion of the y -axis extending from $y = 0$ to $y = -\infty$. In a similar fashion we may determine the remaining portions of the hyperbolas sketched in Figure 76.

We leave as an exercise for the reader the problem of expressing Equation (246) in elliptic coordinates. The result is

$$\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} + k^2(\cosh 2\xi - \cos 2\eta)w = 0, \quad (249)$$

where

$$k = \frac{\omega\gamma}{\sqrt{2}a}. \quad (250)$$

Suppose then that our elliptic membrane has semimajor axis α and semiminor axis β . Its boundary equation is

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$$

(Figure 77). The constant γ of Equation (247) may be chosen so that the membrane is one of the confocal ellipses of Equation (247). For this purpose notice that the distance between the foci of the membrane ellipse is

$$2\sqrt{\alpha^2 - \beta^2},$$

and setting

$$\begin{aligned} \gamma &= \sqrt{\alpha^2 - \beta^2}, \\ \gamma^2 &= \alpha^2 - \beta^2. \end{aligned} \quad (251)$$

Since α and β are given, this equation determines γ .

In order to solve Equation (249) we substitute in it a solution of the form

$$w(\xi, \eta) = \Xi(\xi) H(\eta),$$

and separate the variables, obtaining

$$\frac{\Xi''}{\Xi} + k^2 \cosh 2\xi = -\frac{H''}{H} + k^2 \cos 2\eta.$$

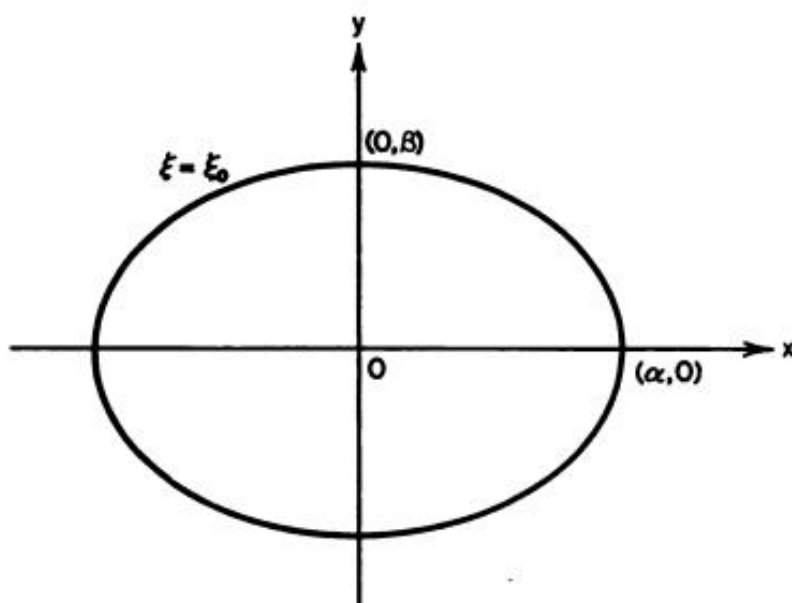


Fig. 77

Calling λ the constant value of the two sides of this equation, we obtain the two ordinary differential equations

$$\begin{aligned} \Xi'' - (\lambda - k^2 \cosh 2\xi)\Xi &= 0 \\ H'' + (\lambda - k^2 \cos 2\eta)H &= 0. \end{aligned} \tag{252}$$

Equations (252) represent another fundamental type of ordinary differential equations known as *Mathieu equations*. In later sections we shall investigate the properties of their solutions and then complete the solution of the boundary value problem for the elliptic membrane.

75. Orthogonal functions

At this point we shall consider certain concepts necessary to a more complete understanding and appreciation of the special

equations derived in this chapter. We shall revert to Fourier series and consider some of their more abstract properties. In this connection it is convenient to relabel the functions $\cos nx$, $n = 0, 1, 2, \dots$ and $\sin nx$, $n = 1, 2, \dots$ as a single infinitude of functions, say, as indicated below :

$$\begin{aligned}\phi_0(x) &= 1. \\ \phi_1(x) &= \cos x. \\ \phi_2(x) &= \sin x. \\ \phi_3(x) &= \cos 2x. \\ \phi_4(x) &= \sin 2x. \\ &\dots \quad \dots \quad \dots \quad \dots \\ \phi_{2n-1}(x) &= \cos nx. \\ \phi_{2n}(x) &= \sin nx. \\ &\dots \quad \dots \quad \dots \quad \dots\end{aligned}$$

Calling the *inner product* of two functions $f(x)$ and $g(x)$, the integral

$$(f, g) \equiv \int_{-\pi}^{\pi} f(x) g(x) dx,$$

the integrals appearing in the Fourier coefficients of $f(x)$ may be indicated in this notation by the symbol

$$(f, \phi_n) \equiv \int_{-\pi}^{\pi} f(x) \phi_n(x) dx.$$

The Fourier series expansion for an arbitrary function $f(x)$ in the interval $(-\pi, \pi)$ then takes the form

$$f(x) = \frac{1}{2\pi}(f, \phi_0)\phi_0(x) + \frac{1}{\pi} \sum_{n=1}^{\infty} (f, \phi_n)\phi_n(x),$$

and the orthogonality conditions for the sine and cosine functions become

$$(\phi_n, \phi_m) = \pi \delta_{nm}, \quad n, m \text{ integers not both zero}, \quad (\phi_0, \phi_0) = 2\pi.$$

We shall show that Legendre, Bessel, and Mathieu functions (solutions of Legendre's, Bessel's, and Mathieu's equations,

respectively) also satisfy orthogonality conditions, and that an arbitrary function $f(x)$ can be expanded in an infinite series of Legendre, Bessel, or Mathieu functions.

We shall first consider some elementary properties of *orthogonal functions* in general. A set of functions

$$\phi_1(x), \phi_2(x), \phi_3(x), \dots$$

is said to be *orthogonal* in the interval $[a, b]$ if

$$\int_a^b \phi_i(x) \phi_j(x) dx = 0, \quad i \neq j,$$

$$\int_a^b \overline{\phi_i(x)}^2 dx \neq 0.$$

If, say,

$$\int_a^b \overline{\phi_i(x)}^2 dx = a_i \neq 0,$$

we can *normalize* the $\phi_i(x)$, that is, find a new set of functions, $\psi_i(x) = \phi_i(x)/\sqrt{a_i}$, which are *orthonormal* in the interval $[a, b]$, that is, such that

$$\int_a^b \psi_i(x) \psi_j(x) dx = \delta_{ij}.$$

In the inner product notation, the set of functions $\{\psi_n(x)\}$ is orthonormal if

$$(\psi_i, \psi_j) = \delta_{ij}.$$

Let us attempt to expand the arbitrary function $f(x)$ in terms of the orthonormal functions $\psi_n(x)$, that is, let us write

$$f(x) = \sum_{n=1}^{\infty} a_n \psi_n(x),$$

where the a_n are undetermined constants. By the condition $(\psi_i, \psi_j) = \delta_{ij}$ we have

$$(f, \psi_k) = (\sum a_n \psi_n, \psi_k) = \sum a_n (\psi_n, \psi_k) = \sum a_n \delta_{nk} = a_k.$$

Hence we may write

$$f(x) = \sum_{k=1}^{\infty} (f, \psi_k) \psi_k(x). \quad (253)$$

So far our work has been purely formal. Does the above equation actually represent an identity? Not always, since, for example, the functions $\sin x$, $\sin 2x$, $\sin 3x$, ... are orthogonal in $(-\pi, \pi)$ and yet we cannot expand $\cos x$ in terms of this orthogonal set. For, if we attempt to, we see that

$$(\cos x, \sin kx) = 0$$

for all k , and we have the absurd result that

$$\cos x \equiv 0.$$

Whenever Equation (253) *does* hold for all values of x , the orthonormal set functions $\{\psi_n(x)\}$ is said to be *complete* over the interval $[a, b]$. Or, more briefly, the $\{\psi_n(x)\}$ form a *complete orthonormal set* over $[a, b]$.

The trigonometric functions $\sin kx$ and $\cos kx$, the complex exponential functions, the Bessel functions, the Legendre functions, and the Mathieu functions are complete orthogonal sets, and therein lies their usefulness: we can expand an arbitrary function in terms of Bessel, Legendre, or Mathieu functions just as we did for sines and cosines. On the other hand, the sine functions are *not* complete in the interval $(-\pi, \pi)$, as shown by the above example; whereas they *are* complete over $(0, \pi)$.

76. The gamma function

As we extend our knowledge of mathematics, certain functions other than the elementary ones appear in our work. For example, we have previously met the error function, $\operatorname{erf} x$ (Section 63) and the sine integral function, $\operatorname{Si}(x)$, (Problem 35, Chapter IV), and we shall soon be compelled to study Legendre, Bessel, and Mathieu functions.

In the definition of Bessel and Legendre functions it is convenient to introduce another new function which can be considered a generalization of factorial. We recall, that if n is a positive integer,

$$n! = n(n-1)(n-2) \cdots 2 \cdot 1$$

while the symbol $\alpha!$ is at present meaningless when α is *not* a positive integer. A function which is defined for all values of the variable α and equals $\alpha!$ when α is a positive integer, is the *gamma function*, $\Gamma(\alpha)$. It is defined by the equation

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx, \quad \alpha > 0. \quad (254)$$

It is easy to show that if α is a positive integer,

$$\Gamma(\alpha) = (\alpha - 1)!.$$

Integrating by parts the integral of Equation (254) :

$$\begin{aligned} \Gamma(\alpha) &= -x^{\alpha-1}e^{-x} \Big|_0^{\infty} + \int_0^{\infty} (\alpha - 1)x^{\alpha-2}e^{-x} dx \\ &= 0 + (\alpha - 1) \int_0^{\infty} x^{\alpha-2}e^{-x} dx \\ &= (\alpha - 1)\Gamma(\alpha - 1), * \alpha > 1. \end{aligned} \quad (255)$$

Applying this result repeatedly, we see that if α is an integer,

$$\Gamma(\alpha - 1) = (\alpha - 2)\Gamma(\alpha - 2),$$

$$\Gamma(\alpha - 2) = (\alpha - 3)\Gamma(\alpha - 3),$$

$$\dots \dots \dots$$

$$\Gamma(3) = 2\Gamma(2),$$

$$\Gamma(2) = 1\Gamma(1),$$

* For any α ,

$$\lim_{x \rightarrow \infty} x^{\alpha-1}e^{-x} = 0.$$

For

$$e^{-x} = \frac{1}{e^x} = \frac{1}{1 + x + \frac{x^2}{2!} + \dots + \frac{x^N}{N!} + \dots}.$$

Now choose N so large that $N > \alpha - 1$. Then for large x ,

$$0 < \frac{x^{\alpha-1}}{e^x} < \frac{x^{\alpha-1}}{\frac{x^N}{N!} + \frac{x^{N+1}}{(N+1)!} + \dots} = \frac{1}{\frac{x^{N-\alpha+1}}{N!} + \frac{x^{N-\alpha+2}}{(N+1)!} + \dots}$$

and since $n - \alpha + 1 > 0$ for all $n \geq N$, every term in the denominator approaches infinity as x approaches infinity, and hence $x^{\alpha-1}e^{-x}$ approaches zero.

from which

$$\Gamma(\alpha) = (\alpha - 1)(\alpha - 2)(\alpha - 3) \cdots 2 \cdot 1 \Gamma(1).$$

The value of $\Gamma(1)$ can be found by direct integration :

$$\Gamma(1) = \int_0^{\infty} x^{1-1} e^{-x} dx = \int_0^{\infty} e^{-x} dx = 1,$$

and hence

$$\Gamma(\alpha) = (\alpha - 1)! \quad \alpha = 1, 2, \dots$$

Equation (255), however, holds for any positive value of the argument and shows that if $\Gamma(\alpha)$ is known for $0 < \alpha \leq 1$, the gamma function can be determined for *any* positive value of α . For if β is any positive number, we may always write

$$\beta = m + \alpha,$$

where m is an integer and $0 < \alpha \leq 1$. If $\alpha = 1$, then

$$\Gamma(\beta) = m!$$

since β is an integer. If $\alpha \neq 1$, then by Equation (255),

$$\Gamma(\beta) = (\beta - 1)(\beta - 2) \cdots (\beta - m)\Gamma(\alpha),$$

which is readily computed when $\Gamma(\alpha)$ is known. For the values of $\Gamma(\alpha)$ see [14] and Figure 78.

So far we have defined $\Gamma(\alpha)$ only for $\alpha > 0$. If α is negative we define $\Gamma(\alpha)$ by means of Equation (255) as

$$\Gamma(\alpha) = \frac{\Gamma(\alpha + 1)}{\alpha} \quad -(N + 1) < \alpha < -N \quad (256)$$

for $N = 0, 1, 2, \dots$ For example,

$$\begin{aligned} \Gamma(-3.2) &= \frac{\Gamma(-2.2)}{-3.2} = \frac{1}{-3.2} \left[\frac{\Gamma(-1.2)}{-2.2} \right] = \frac{1}{(3.2)(2.2)} \left[\frac{\Gamma(-.2)}{-1.2} \right] \\ &= \frac{-1}{(3.2)(2.2)(1.2)} \left[\frac{\Gamma(0.8)}{-0.2} \right] = \frac{1}{(3.2)(2.2)(1.2)(0.2)} \Gamma(0.8) \\ &= 0.6891. \end{aligned}$$

As an immediate corollary of the above definition,

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1) \quad \alpha \neq 0, -1, -2, \dots$$

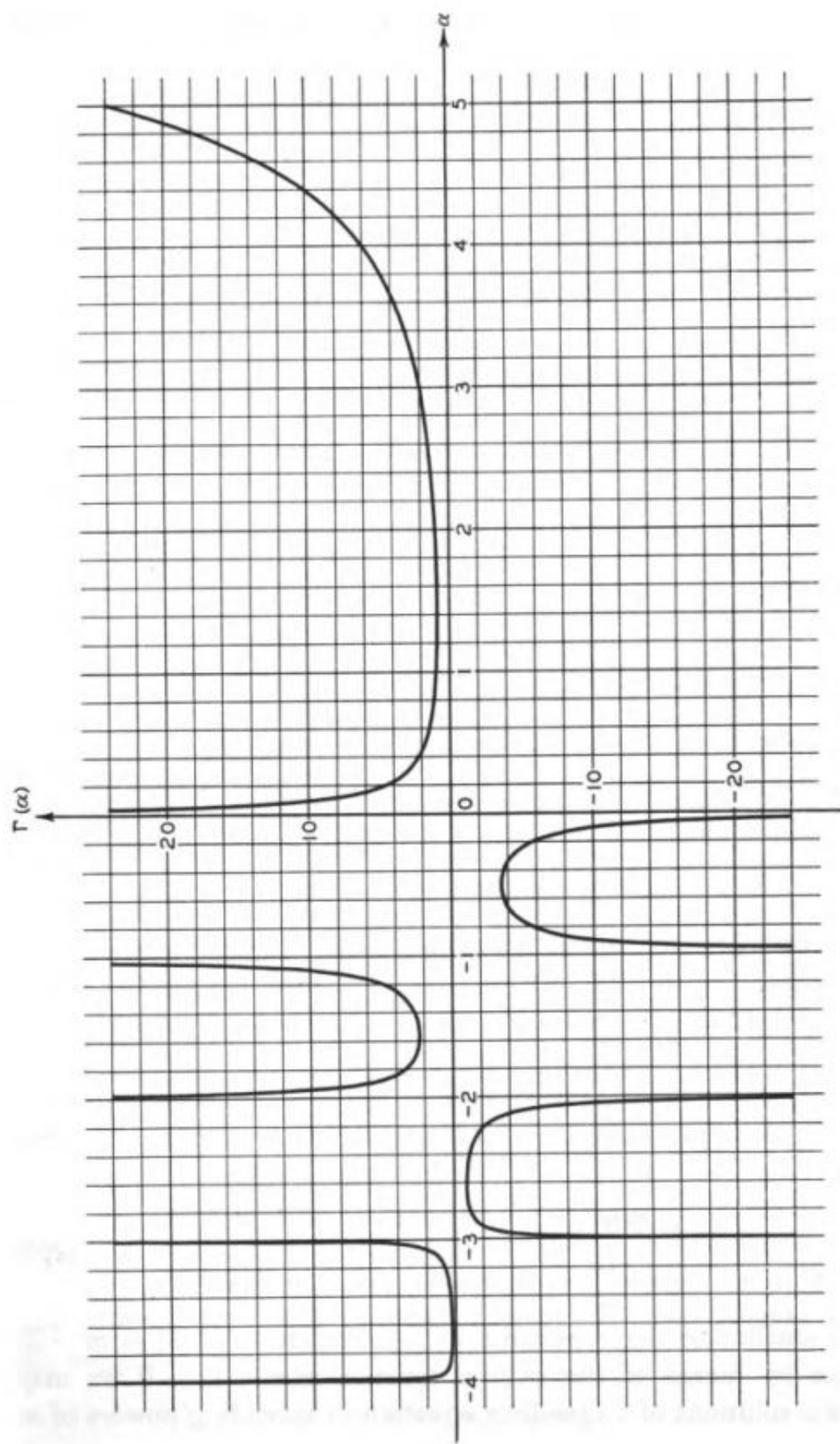


Fig. 78

When $\alpha = 0$ (or a negative integer), the gamma function becomes infinite (see Figure 78). This can be shown analytically by writing Equation (256) with $\alpha = \epsilon > 0$,

$$\Gamma(\epsilon) = \frac{\Gamma(\epsilon + 1)}{\epsilon}$$

and then taking the limit as $\epsilon \rightarrow 0$. As $\epsilon \rightarrow 0$, $\Gamma(\epsilon + 1) \rightarrow \Gamma(1) = 1$ and

$$\Gamma(0) = \lim_{\epsilon \rightarrow 0} \Gamma(\epsilon) = \lim_{\epsilon \rightarrow 0} \frac{\Gamma(\epsilon + 1)}{\epsilon} = \infty.$$

For negative integral values of N , Equation (256), applied repeatedly, gives

$$\begin{aligned}\Gamma(-N) &= \frac{1}{(-N)(-N+1)(-N+2)\dots(-1)} \Gamma(0) \\ &= \frac{(-1)^N}{N!} \Gamma(0) = \pm\infty.\end{aligned}$$

77. Solution of Legendre's equation

We have called Legendre's equation in Section 72 the equation [Equation (240)] :

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0. \quad (257)$$

This ordinary differential equation may be solved by the method of Frobenius (see [25], Chapter VII). Substituting $y = x^m$ into Equation (257), we obtain

$$\begin{aligned}m(m-1)x^{m-2} - m(m-1)x^m - 2mx^m + n(n+1)x^m \\ = [m(m-1)]x^{m-2} + [-m^2 - m + n(n+1)]x^m\end{aligned} \quad (258)$$

and

$$m(m-1) = 0 \text{ implies } m = 0, 1, \quad (259)$$

$$m^2 + m - n(n+1) = 0 \text{ implies } m = n, -(n+1).$$

The smaller of the exponents of x in Equation (258) is $m-2$; hence by means of the *indicial equation* $m(m-1) = 0$ we may obtain solutions of Legendre's equation in ascending powers of x .

Since one of the roots of $m(m-1) = 0$ is zero, we may assume a solution of the form

$$y(x) = c_0 + c_2 x^2 + c_4 x^4 + \dots \equiv \sum_{k=0}^{\infty} c_{2k} x^{2k}. \quad (260)$$

If we substitute this in Equation (257), there results

$$\begin{aligned} \sum_{k=0}^{\infty} 2k(2k-1)c_{2k}x^{2k-2} - \sum_{k=0}^{\infty} 2k(2k-1)c_{2k}x^{2k} - 2 \sum_{k=0}^{\infty} 2kc_{2k}x^{2k} \\ + n(n+1) \sum_{k=0}^{\infty} c_{2k}x^{2k} = 0, \end{aligned}$$

or collecting powers of x :

$$\begin{aligned} \sum_{k=0}^{\infty} [(2k+2)(2k+1)c_{2k+2} - 2k(2k-1)c_{2k} \\ - 2(2k)c_{2k} + n(n+1)c_{2k}]x^{2k} = 0. \quad (261) \end{aligned}$$

If Equation (260) is to satisfy Legendre's equation, the coefficients of all powers of x in Equation (261) must vanish, and hence

$$\begin{aligned} 2c_2 + n(n+1)c_0 &= 0 && \text{coefficient of } x^0, \\ 4 \cdot 3c_4 - 2 \cdot 1c_2 - 2 \cdot 2c_2 + n(n+1)c_2 &= 0 && \text{coefficient of } x^2, \\ 6 \cdot 5c_6 - 4 \cdot 3c_4 - 2 \cdot 4c_4 + n(n+1)c_4 &= 0 && \text{coefficient of } x^4, \\ \dots &&& \dots \end{aligned}$$

From the equation for the coefficient of x^0 we obtain

$$c_2 = -\frac{n(n+1)}{2!} c_0. \quad (262)$$

The second equation gives

$$c_4 = \frac{[6 - n(n+1)]}{4 \cdot 3} c_2 = \frac{(n-2)n(n+1)(n+3)}{4!} c_0,$$

and proceeding in this fashion we obtain all the c_{2k} and hence the integral of Legendre's equation:

$$y(x) = c_0 y_1(x),$$

where

$$y_1(x) = 1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 \\ - \frac{(n-4)(n-2)n(n+1)(n+3)(n+5)}{6!}x^6 + \dots \quad (263)$$

Since the difference of the roots of the equation $m(m-1)=0$ is not a multiple of the difference of the exponents of x of Equation (258), we obtain another linearly independent solution

$$y(x) = c_1x + c_3x^3 + c_5x^5 + \dots \equiv \sum_{k=0}^{\infty} c_{2k+1}x^{2k+1},$$

corresponding to the root $m=1$. [If the second root had been a multiple of 2, this would be the same solution as Equation (260) since it would then involve only even powers.] Substituting the above series into Equation (257) and equating coefficient of powers of x to zero we obtain

$$y(x) = c_1y_2(x),$$

where

$$y_2(x) = x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 \\ - \frac{(n-5)(n-3)(n-1)(n+2)(n+4)(n+6)}{7!}x^7 \\ + \dots \quad (264)$$

Using the test-ratio test, [24], Section 5.10, we can show that the power series $y_1(x)$ and $y_2(x)$ are convergent for $|x| < 1$. Hence if $|x| < 1$, the general solution of Legendre's equation is given by

$$y(x) = Ay_1(x) + By_2(x),$$

where A and B are arbitrary constants and $y_1(x)$ and $y_2(x)$ are given by Equations (263) and (264), respectively.

Turning now to the second indicial equation $m^2+m-n(n+1)=0$, we assume first a solution of the form

$$y(x) = a_0x^n + a_1x^{n-2} + a_2x^{n-4} + \dots$$

in *descending* powers of x corresponding to the root $m = n$, while the root $m = -n - 1$ leads to a solution of the form

$$y(x) = b_0 x^{-n-1} + b_1 x^{-n-3} + b_2 x^{-n-5} + \dots$$

Equating the coefficients of all the powers of x to zero, we determine as before two linearly independent solutions of Legendre's equation :

$$\begin{aligned} y_1^*(x) = x^n &- \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2^2 2!(2n-1)(2n-3)} x^{n-4} \\ &- \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{2^3 3!(2n-1)(2n-3)(2n-5)} x^{n-6} \\ &+ \dots \end{aligned} \quad (265)$$

$$\begin{aligned} y_2^*(x) = x^{-n-1} &+ \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} \\ &+ \frac{(n+1)(n+2)(n+3)(n+4)}{2^2 2!(2n+3)(2n+5)} x^{-n-5} \\ &+ \frac{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}{2^3 3!(2n+3)(2n+5)(2n+7)} x^{-n-7} \\ &+ \dots \end{aligned} \quad (266)$$

These two power series may be shown to be convergent for $|x| > 1$. Hence

$$y(x) = Cy_1^*(x) + Dy_2^*(x) \quad (267)$$

is the general solution of Legendre's equation for $|x| > 1$ where C, D are arbitrary constants, and $y_1^*(x), y_2^*(x)$ are given by Equations (265) and (266), respectively.

A careful analysis of Equations (263), (264), (265), and (266) shows that the functions y_1, y_2, y_1^*, y_2^* are related to one another when n is an integer. In particular it may be shown that

(a) If n is a nonnegative integer,

$$y_1^*(x) = k_1 y_1(x),$$

if n is even, and

$$y_1^*(x) = k_2 y_2(x)$$

if n is odd, where k_1 and k_2 are nonzero constants.

(b) If n is a negative integer,

$$y_1^*(x) = k_3 y_1(x)$$

if n is odd, and

$$y_2^*(x) = k_4 y_2(x)$$

if n is even, where k_3 and k_4 are nonzero constants.

We see therefore that if n is a nonnegative integer, the series defined by $y_1^*(x)$ is *finite*, since after a certain term each numerator in the coefficients of the powers of x will contain a zero. These finite solutions are essentially the so-called *Legendre polynomials*. It is customary to define the Legendre polynomials, $P_n(x)$ for $n = 0, 1, 2, \dots$ by the equation

$$P_n(x) = \frac{(2n)!}{2^n(n!)^2} y_1^*(x). \quad (268)$$

These polynomials are also called *zonal harmonics*. The functions $Q_n(x)$ defined by

$$Q_n(x) = \frac{2^n(n!)^2}{(2n+1)!} y_2^*(x)$$

are sometimes called the *Legendre functions of the second kind* or *zonal harmonics of the second kind*.*

We leave as an exercise for the reader the task of verifying that the first few Legendre polynomials are given by :

$$\begin{aligned} P_0(x) &= 1. \\ P_1(x) &= x. \\ P_2(x) &= \frac{3}{2}x^2 - \frac{1}{2}. \\ P_3(x) &= \frac{5}{2}x^3 - \frac{3}{2}x. \\ P_4(x) &= \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8}. \\ &\dots \dots \dots \end{aligned} \quad (269)$$

* If n is a nonnegative integer the general solution of Legendre's equations for $|x| > 1$ is given by

$$y(x) = C_1 P_n(x) + C_2 Q_n(x),$$

where C_1 and C_2 are arbitrary constants. If $|x| < 1$, then the $P_n(x)$ are still solutions of Legendre's equation, but the $Q_n(x)$ must be replaced by $y_1(x)$ or $y_2(x)$, depending on whether n is odd or even.

78. Orthogonality of the Legendre polynomials

The usefulness of the Legendre polynomials lies in the fact that they form a complete orthogonal set (see Section 75), so that arbitrary functions can be expanded in terms of the Legendre polynomials. We shall show in this section that the functions $\{P_n(x)\}$ are orthogonal in the interval $-1 \leq x \leq 1$, and that

$$\int_{-1}^1 P_m(x)P_n(x) dx = 0 \quad m \neq n \quad n, m = 0, 1, 2, \dots, \quad (270)$$

$$\int_{-1}^1 P_n^2(x) dx = \frac{2}{2n+1}, \quad n = 0, 1, 2, \dots *$$

To prove the orthogonality conditions, recall that if $P_n(x)$ and $P_m(x)$ are two Legendre polynomials they satisfy Legendre's equation,

$$\frac{d}{dx} \left[(1-x^2) \frac{d}{dx} P_n \right] + n(n+1)P_n = 0,$$

and

$$\frac{d}{dx} \left[(1-x^2) \frac{d}{dx} P_m \right] + m(m+1)P_m = 0,$$

respectively. Multiplying the first of these equations of P_m , the second by P_n , and subtracting, we obtain

$$P_n \frac{d}{dx} [(1-x^2)P'_m] - P_m \frac{d}{dx} [(1-x^2)P'_n] + [m(m+1) - n(n+1)]P_n P_m = 0. \quad (271)$$

But upon integrating by parts,

$$\begin{aligned} \int_{-1}^1 P_n \frac{d}{dx} [(1-x^2)P'_m] dx &= P_n (1-x^2)P'_m \Big|_{-1}^{+1} - \int_{-1}^1 (1-x^2)P'_m P'_n dx \\ &= - \int_{-1}^1 (1-x^2)P'_m P'_n dx, \end{aligned}$$

* If we define

$$p_n(x) = \sqrt{\frac{2n+1}{2}} P_n(x)$$

the set of functions $\{p_n(x)\}$ will form an *orthonormal* set in $[-1, 1]$, and we may write

$$\int_{-1}^1 p_n(x)p_m(x)dx = \delta_{nm}, \quad n, m = 0, 1, 2, \dots$$

and integration of Equation (271) yields

$$- \int_{-1}^1 (1-x^2)P'_m P'_n dx + \int_{-1}^1 (1-x^2)P'_n P'_m dx \\ + (m^2 + m - n^2 - n) \int_{-1}^1 P_m P_n dx = 0.$$

If we assume that $m \neq n$ and hence $m^2 + m \neq n^2 + n$,

$$\int_{-1}^1 P_m(x)P_n(x) dx = 0, \quad m \neq n.$$

To prove the second of the orthogonality conditions, we assume known a property of the Legendre functions which is proved in the next section, namely,

$$\frac{1}{\sqrt{1-2\lambda x + \lambda^2}} = \sum_{n=0}^{\infty} \lambda^n P_n(x). \quad (272)$$

If we write Equation (272) for another value of λ , say μ :

$$\frac{1}{\sqrt{1-2\mu x + \mu^2}} = \sum_{m=0}^{\infty} \mu^m P_m(x),$$

and multiply this by Equation (272) we have, on integrating from -1 to $+1$:

$$\int_{-1}^1 \frac{dx}{\sqrt{(1-2\lambda x + \lambda^2)(1-2\mu x + \mu^2)}} = \\ \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \lambda^n \mu^m \int_{-1}^1 P_n(x)P_m(x) dx. \quad (273)$$

The integral on the left-hand side of this equation is (see [22], Formula 113)

$$\frac{1}{\sqrt{\lambda\mu}} \log \frac{1 + \sqrt{\lambda\mu}}{1 - \sqrt{\lambda\mu}}.$$

If we expand the above function in powers of $\lambda\mu$,

$$\frac{1}{\sqrt{\lambda\mu}} \log \frac{1 + \sqrt{\lambda\mu}}{1 - \sqrt{\lambda\mu}} = \frac{1}{\sqrt{\lambda\mu}} [\log(1 + \sqrt{\lambda\mu}) - \log(1 - \sqrt{\lambda\mu})] \\ = 2 \left(1 + \frac{\lambda\mu}{3} + \frac{\lambda^2\mu^2}{5} + \dots + \frac{\lambda^k\mu^k}{2k+1} + \dots \right)$$

and equate powers of λ^μ in Equation (273) we see that

$$\int_{-1}^1 P_n^2(x) dx = \frac{2}{2n+1}.*$$

We have shown that

$$\int_{-1}^1 P_n(x)P_m(x) dx = \frac{2}{2n+1}\delta_{nm}. \quad (274)$$

The orthogonality of the Legendre functions is therefore established. The completeness of the $\{P_n(x)\}$ will not be proved.

79. Generating function for the Legendre polynomials

Consider the function

$$F(\lambda, x) = \frac{1}{\sqrt{1-2\lambda x + \lambda^2}},$$

which arises in the mathematical theory of potential. If we expand it in a power series in λ ,

$$F(\lambda, x) = \sum_{n=0}^{\infty} \lambda^n R_n(x), \quad (275)$$

where the coefficients of the powers of λ are functions of x . We shall prove by expanding $F(\lambda, x)$ by the binomial theorem and then equating powers of λ in Equation (275) that the $R_n(x)$ are precisely the Legendre polynomials $P_n(x)$. The binomial expansion of $F(\lambda, x)$ is given by

$$\begin{aligned} F(\lambda, x) &= 1 - \frac{1}{2}(\lambda^2 - 2\lambda x) + \frac{1}{2}\left(\frac{1 \cdot 3}{2 \cdot 2}\right)(\lambda^2 - 2\lambda x)^2 \\ &\quad - \frac{1}{3 \cdot 2}\left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 2 \cdot 2}\right)(\lambda^2 - 2\lambda x)^3 + \dots \\ &\quad + \frac{(-1)^n}{n!} \frac{[1 \cdot 3 \cdot 5 \dots (2n-1)]}{2^n} (\lambda^2 - 2\lambda x)^n \\ &\quad - \dots \end{aligned}$$

* From the power series expansion we could also conclude that

$$\int_{-1}^1 P_n(x)P_m(x)dx = 0, \quad n \neq m,$$

since the coefficient of $\lambda^n \mu^m$ ($n \neq m$) on the left-hand side of Equation (273) is zero.

Given an integer β , every term of this expansion for which $\beta \geq n \geq \beta/2$ contains the term λ^β . For example, the β 'th term of the above expansion may be written

$$\frac{(-1)^\beta}{\beta!} \frac{[1 \cdot 3 \cdot 5 \dots (2\beta - 1)]}{2^\beta} \lambda^\beta \sum_{k=0}^{\beta} \binom{\beta}{k} (2x)^{\beta-k} (-1)^{\beta-k} \lambda^k, *$$

and the coefficient of λ^β in it is

$$\frac{(-1)^\beta}{\beta!} \frac{[1 \cdot 3 \cdot 5 \dots (2\beta - 1)]}{2^\beta} (-1)^\beta (2x)^\beta = \frac{1 \cdot 3 \cdot 5 \dots (2\beta - 1)}{\beta!} x^\beta.$$

Similarly the $(\beta - 1)$ st term is

$$\frac{(-1)^{\beta-1}}{(\beta-1)!} \frac{[1 \cdot 3 \cdot 5 \dots (2\beta - 3)]}{2^{\beta-1}} \lambda^{\beta-1} \sum_{k=0}^{\beta-1} \binom{\beta-1}{k} (2x)^{\beta-1-k} (-1)^{\beta-1-k} \lambda^k,$$

and the coefficient of λ^β in this expression is

$$\begin{aligned} \frac{(-1)^{\beta-1}}{(\beta-1)!} \frac{[1 \cdot 3 \cdot 5 \dots (2\beta - 3)]}{2^{\beta-1}} \binom{\beta-1}{1} (2x)^{\beta-2} (-1)^{\beta-2} \\ = - \frac{1 \cdot 3 \cdot 5 \dots (2\beta - 3)}{2(\beta-2)!} x^{\beta-2}. \end{aligned}$$

The coefficient of λ^β in the $(\beta - 2)$ nd term may similarly be shown to be

$$\begin{aligned} \frac{(-1)^{\beta-2}}{(\beta-2)!} \frac{[1 \cdot 3 \cdot 5 \dots (2\beta - 5)]}{2^{\beta-2}} \binom{\beta-2}{2} (2x)^{\beta-4} (-1)^{\beta-4} \\ = - \frac{1 \cdot 3 \cdot 5 \dots (2\beta - 5)}{2^2 2! (\beta-4)!} x^{\beta-4}. \end{aligned}$$

Continuing in this fashion we find that the *total* coefficient of λ^β in the expansion of $F(\lambda, x)$ is

$$\begin{aligned} R_\beta(x) = \frac{1 \cdot 3 \cdot 5 \dots (2\beta - 1)}{\beta!} \left[x^\beta - \frac{\beta(\beta-1)}{2(2\beta-1)} x^{\beta-2} \right. \\ \left. + \frac{\beta(\beta-1)(\beta-2)(\beta-3)}{2^2 2! (2\beta-1)(2\beta-3)} x^{\beta-4} - \dots \right], \end{aligned}$$

and comparing this with $y^*_1(x)$ of Equation (268) shows that

$$R_\beta(x) = P_\beta(x).$$

$$* \binom{\beta}{k} \text{ is the binomial coefficient: } \binom{\beta}{k} = \frac{\beta!}{k!(\beta-k)!}.$$

80. Completion of the solution of Section 72

We considered in Section 72 the problem of the potential in a sphere which led to the partial differential equation

$$\sin \theta \frac{\partial}{\partial r} \left(r^2 \frac{\partial u}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial u}{\partial \theta} \right) = 0. \quad (276)$$

By means of the substitution

$$u(r, \theta) = R(r)\Theta(\theta)$$

we reduced Equation (276) to the two ordinary differential equations

$$r^2 R'' + 2rR' - n(n+1)R = 0$$

$$\frac{d}{dx} \left[(1-x^2) \frac{d\Theta}{dx} \right] + n(n+1)\Theta = 0, \quad x = \cos \theta.$$

The first we recognized as a Cauchy equation, which reduces to

$$R'' + R' - n(n+1)R = 0$$

by means of the substitution $r = e^t$. Its general solution is

$$R(r) = C_1 e^{nt} + C_2 e^{-(n+1)t}$$

or, in terms of r ,

$$R(r) = C_1 r^n + C_2 r^{-(n+1)}.$$

The second equation is a Legendre equation whose general solution is given by

$$\Theta(\theta) = C_3 y_1(\cos \theta) + C_4 y_2(\cos \theta).$$

[We use the solutions $y(x)$, $|x| < 1$ since $|\cos \theta| \leq 1$.] When $\theta = 0$ or π , (the z -axis), $|\cos \theta| = 1$ and the only solutions of Legendre's equation which are finite for $|\cos \theta| = 1$ are the Legendre *polynomials* $P_n(\cos \theta)$. Also as $r \rightarrow 0$, $r^{-(n+1)} \rightarrow \infty$, which is physically unreasonable. Hence the only solutions which remain finite are of the form

$$u_n(r, \theta) = r^n P_n(\cos \theta).$$

Any linear combination of these u_n is a solution of Equation (276), and hence the function

$$u(r, \theta) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos \theta) \quad (277)$$

satisfies the partial differential equation, whatever the values of the constants a_n . The a_n can be determined from the boundary condition giving the potential distribution on the surface of the sphere,

$$u(A, \theta) = f(\theta),$$

which we find convenient to write in the form

$$f(\theta) \equiv g(\cos \theta).$$

By means of Equation (277) this condition gives

$$u(A, \theta) = g(\cos \theta) = \sum_{n=0}^{\infty} a_n A^n P_n(\cos \theta).$$

Due to the orthogonality of the Legendre polynomials, we have, on multiplying both sides of this equation by $P_m(\cos \theta)$ and integrating from -1 to $+1$,*

$$\int_{-1}^1 g(z) P_m(z) dz = A^m a_m \int_{-1}^1 P_m^2(z) dz = \frac{2A^m a_m}{2m+1},$$

or

$$a_m = \frac{2m+1}{2A^m} \int_{-1}^1 g(z) P_m(z) dz.$$

The final solution of the boundary value problem is therefore

$$u(r, \theta) = \frac{1}{2} \sum_{n=0}^{\infty} \left[(2n+1) \int_{-1}^1 g(z) P_n(z) dz \right] \left(\frac{r}{A} \right)^n P_n(\cos \theta). \quad (278)$$

As an example, suppose that the initial potential distribution is

$$f(\theta) = g(\cos \theta) = \sin(\pi \cos \theta).$$

Then

$$g(z) = \sin \pi z,$$

* For simplicity we have set $\cos \theta \equiv z$.

and

$$\begin{aligned}
 a_0 &= \frac{1}{2} \int_{-1}^1 P_0(z) \sin \pi z \, dz = \frac{1}{2} \int_{-1}^1 \sin \pi z \, dz = 0, \\
 a_1 &= \frac{3}{2A} \int_{-1}^1 P_1(z) \sin \pi z \, dz = \frac{3}{2A} \int_{-1}^1 z \sin \pi z \, dz = \frac{3}{\pi A}, \\
 a_2 &= \frac{5}{2A^2} \int_{-1}^1 P_2(z) \sin \pi z \, dz = \frac{5}{2A^2} \int_{-1}^1 \left[\frac{3}{2} z^2 - \frac{1}{2} \right] \sin \pi z \, dz = 0, \\
 a_3 &= \frac{7}{2A^3} \int_{-1}^1 P_3(z) \sin \pi z \, dz = \frac{7}{2A^3} \int_{-1}^1 \left[\frac{5}{2} z^3 - \frac{3}{2} z \right] \sin \pi z \, dz, \\
 &= \frac{7}{\pi A^3} - \frac{105}{\pi^3 A^3}.
 \end{aligned}$$

... ..

Hence, by Equation (277),

$$u(r, \theta) = \frac{3}{\pi} \frac{r}{A} P_1(\cos \theta) - \left(\frac{105}{\pi^3} - \frac{7}{\pi} \right) \frac{r^3}{A^3} P_3(\cos \theta) + \dots$$

81. Solution of Bessel's equation

If we write Bessel's equation (Section 73) in the form

$$x^2 y'' + xy' + (x^2 - n^2)y = 0 \quad (279)$$

we find by the method of Frobenius two linearly independent solutions

$$\begin{aligned}
 y_1(x) = x^n &\left[1 - \frac{x^2}{2^2(n+1)} + \frac{x^4}{2^4 \cdot 2!(n+1)(n+2)} \right. \\
 &\quad \left. - \frac{x^6}{2^6 \cdot 3!(n+1)(n+2)(n+3)} + \dots \right]
 \end{aligned}$$

and

$$\begin{aligned}
 y_2(x) = x^{-n} &\left[1 - \frac{x^2}{2^2(1-n)} + \frac{x^4}{2^4 \cdot 2!(1-n)(2-n)} \right. \\
 &\quad \left. - \frac{x^6}{2^6 \cdot 3!(1-n)(2-n)(3-n)} + \dots \right].
 \end{aligned}$$

If n is *not* an integer, both $y_1(x)$ and $y_2(x)$ converge for all values of x . If n is a positive integer, $y_1(x)$ converges but $y_2(x)$ does not; if n is a negative integer, $y_2(x)$ converges but $y_1(x)$ does

not. Finally, if $n = 0$, $y_1(x) = y_2(x)$ converges for all values of x . We thus see that if n is an integer only one of the above functions is a solution. The other linearly independent solution corresponding to $y_1(x)$ may be proved to be (see [25], Chapter VIII)

$$y_1(x) \int \frac{dx}{xy_1^2(x)}, \quad (280)$$

with a similar expression corresponding to $y_2(x)$.

The function

$$J_n(x) = \frac{y_1(x)}{2^n \Gamma(n+1)}$$

is called the *Bessel function of the first kind of order n* . The function

$$J_{-n}(x) = \frac{y_2(x)}{2^{-n} \Gamma(1-n)} \quad (281)$$

is called the Bessel function of the first kind of order $-n$, and is linearly independent of $J_n(x)$ if n is not an integer. We see, therefore, that if n is *not* an integer, the general solution of Bessel's equation may be written

$$y(x) = C_1 J_n(x) + C_2 J_{-n}(x),$$

where C_1 and C_2 are arbitrary constants. If n is an integer, the second linearly independent solution of Bessel's equation [Equation (280)] is known as the *Bessel function of the second kind of order n* and is generally indicated by $Y_n(x)$. Hence if n is an integer, the general solution of Bessel's equation may be written

$$y(x) = C_3 J_n(x) + C_4 Y_n(x), \quad (282)$$

where C_3 and C_4 are arbitrary constants.

82. Properties of the Bessel functions

We shall obtain in this section some properties of Bessel functions necessary to prove their orthogonality (see the next section). Other identities and relations are left to the problems (see end of chapter).

With the aid of the gamma function, the Bessel functions of the first kind of order $\pm n$ may be written in the compact form

$$J_n(x) = \sum_{\alpha=0}^{\infty} \frac{(-1)^\alpha}{\alpha! \Gamma(n + \alpha + 1)} \left(\frac{x}{2}\right)^{n+2\alpha}, \quad (283)$$

$$J_{-n}(x) = \sum_{\alpha=0}^{\infty} \frac{(-1)^\alpha}{\alpha! \Gamma(-n + \alpha + 1)} \left(\frac{x}{2}\right)^{-n+2\alpha}. \quad (284)$$

When n is a positive integer we have (see Section 76)

$$\Gamma(-n + \alpha + 1) = \infty \quad \text{for } \alpha = 0, 1, \dots, n-1.$$

Hence

$$\begin{aligned} J_{-n}(x) &= \sum_{\alpha=n}^{\infty} \frac{(-1)^\alpha}{\alpha! \Gamma(-n + \alpha + 1)} \left(\frac{x}{2}\right)^{-n+2\alpha} \\ &= \sum_{\alpha=n}^{\infty} \frac{(-1)^\alpha}{\alpha! (\alpha - n)!} \left(\frac{x}{2}\right)^{-n+2\alpha} = \sum_{\beta=0}^{\infty} \frac{(-1)^{\beta+n}}{(\beta + n)! \beta!} \left(\frac{x}{2}\right)^{2\beta+n} \end{aligned}$$

on changing the dummy index of summation from α to $\beta = \alpha - n$. But if $n = 0, 1, 2, 3, \dots$, then $J_n(x)$ as given by Equation (283) becomes

$$J_n(x) = \sum_{\alpha=0}^{\infty} \frac{(-1)^\alpha}{\alpha! (n + \alpha)!} \left(\frac{x}{2}\right)^{n+2\alpha}$$

Hence

$$J_{-n}(x) = (-1)^n J_n(x), \quad n = 0, 1, 2, 3, \dots \quad (285)$$

Another formula to be used in Section 83 is obtained by multiplying Equation (283) by x^n and then differentiating with respect to x :

$$\begin{aligned} \frac{d}{dx} [x^n J_n(x)] &= \frac{d}{dx} \sum_{\alpha=0}^{\infty} \frac{(-1)^\alpha}{\alpha! \Gamma(n + \alpha + 1)} \frac{x^{2n+2\alpha}}{2^{n+2\alpha}} \\ &= x^n \sum_{\alpha=0}^{\infty} \frac{(-1)^\alpha (n + \alpha)}{\alpha! \Gamma(n + \alpha + 1)} \left(\frac{x}{2}\right)^{n+2\alpha-1} \end{aligned}$$

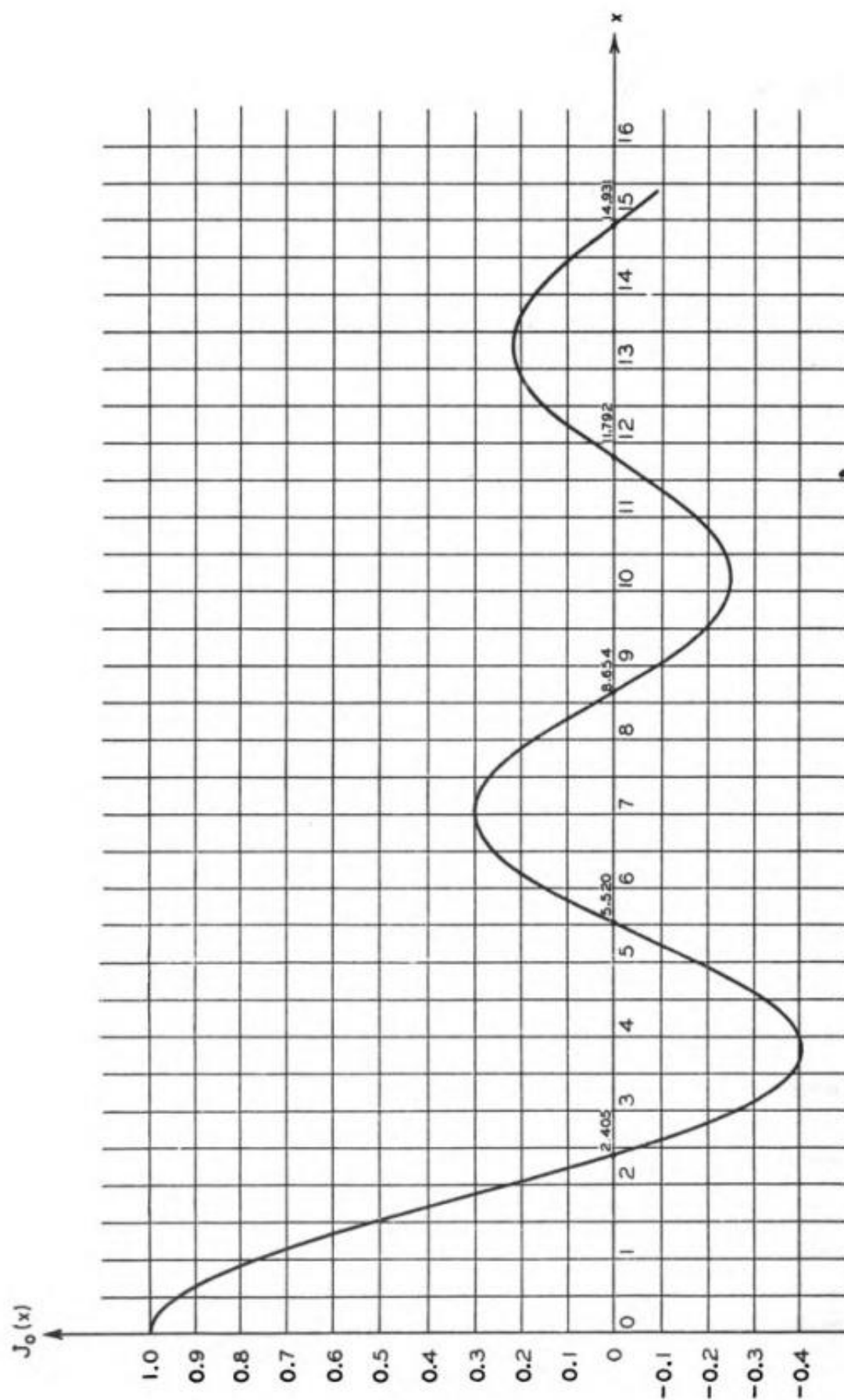


Fig. 79

But if we replace n by $n - 1$ in Equation (283),

$$\begin{aligned} J_{n-1}(x) &= \sum_{\alpha=0}^{\infty} \frac{(-1)^{\alpha}}{\alpha! \Gamma(n+\alpha)} \left(\frac{x}{2}\right)^{n+2\alpha-1} \\ &= \sum_{\alpha=0}^{\infty} \frac{(-1)^{\alpha} (n+\alpha)}{\alpha! \Gamma(n+\alpha+1)} \left(\frac{x}{2}\right)^{n+2\alpha-1}. \end{aligned}$$

Hence

$$\frac{d}{dx} [x^n J_n(x)] = x^n J_{n-1}(x). \quad (286)$$

The Bessel function most frequently encountered in the applications is $J_0(x)$. It is convenient to write out explicitly its power series expansion :

$$\begin{aligned} J_0(x) &= \sum_{\alpha=0}^{\infty} \frac{(-1)^{\alpha}}{(\alpha!)^2} \left(\frac{x}{2}\right)^{2\alpha} \\ &= 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots \end{aligned} \quad (287)$$

Its graph has the appearance of a damped cosine wave. [See Figure 79 for an accurate plot of $J_0(x)$.] The other linearly independent solution $Y_0(x)$ of Bessel's equation of order zero is sketched in Figure 80.

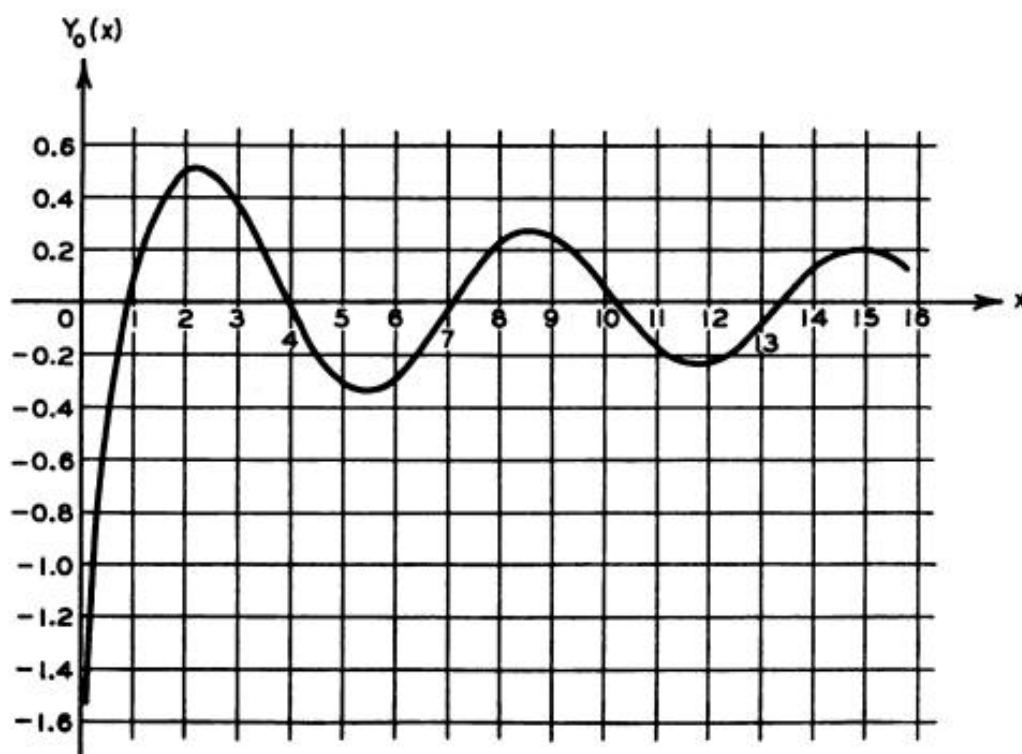


Fig. 80

83. Orthogonality of the Bessel functions

It is clear from the graph of Figure 79 that $J_0(x)$ has infinitely many positive roots, which we shall denote by $\lambda_1, \lambda_2, \lambda_3, \dots$ where $0 < \lambda_1 < \lambda_2 < \dots$. We shall now show that the functions

$$\left\{ \sqrt{x} J_0\left(\lambda_i \frac{x}{c}\right) \right\} \quad i = 1, 2, 3, \dots$$

are orthogonal in the interval $[0, c]$ and more particularly that

$$\int_0^c x J_0\left(\lambda_i \frac{x}{c}\right) J_0\left(\lambda_j \frac{x}{c}\right) dx = \frac{c^3}{2} [J_1(\lambda_i)]^2 \delta_{ij}.^* \quad (288)$$

Before starting the proof of Equation (288) we remark that the above statements can be generalized to Bessel functions of the first kind of all orders $n \geq 0$, viz.,

(a) $J_n(x)$ has infinitely many positive roots, say

$$0 < \lambda_1^{(n)} < \lambda_2^{(n)} < \lambda_3^{(n)} < \dots$$

(b) The functions

$$J_n\left(\lambda_i^{(n)} \frac{x}{c}\right)$$

are orthogonal in the interval $[0, c]$ with respect to the weighting function x , and

$$\int_0^c x J_n\left(\lambda_i^{(n)} \frac{x}{c}\right) J_n\left(\lambda_j^{(n)} \frac{x}{c}\right) dx = \frac{c^3}{2} [J_{n+1}(\lambda_i^{(n)})]^2 \delta_{ij}.$$

To prove the orthogonality of the J_0 functions it is convenient to introduce the functions

$$u(x) = \sqrt{x} J_0(\alpha x), \quad v(x) = \sqrt{x} J_0(\beta x), \quad (289)$$

* To say that the functions $\left\{ \sqrt{x} J_0\left(\lambda_i \frac{x}{c}\right) \right\}$ are orthogonal in the interval $[0, c]$ is equivalent to saying that the functions $J_0\left(\lambda_i \frac{x}{c}\right)$ are orthogonal in the interval $[0, c]$ with respect to the weighting function x .

where α and β are two constants. Since $J_0(x)$ satisfies Bessel's equation (Equation 279 with $n = 0$), $J_0(\alpha x)$ must satisfy the equation

$$\begin{aligned} (\alpha x)^2 \frac{d^2}{d(\alpha x)^2} J_0(\alpha x) + (\alpha x) \frac{d}{d(\alpha x)} J_0(\alpha x) + (\alpha x)^2 J_0(\alpha x) \\ = x^2 \frac{d^2}{dx^2} J_0(\alpha x) + x \frac{d}{dx} J_0(\alpha x) + \alpha^2 x^2 J_0(\alpha x) = 0. \end{aligned}$$

Substituting in this equation the value of $J_0(\alpha x)$ as given by Equation (289), we obtain

$$x^2(x^{-1/2}u)'' + x(x^{-1/2}u)' + \alpha^2 x^2(x^{-1/2}u) = 0,$$

or

$$4x^2 u'' + (4\alpha^2 x^2 + 1)u = 0. \quad (290)$$

It may be similarly proved that v satisfies the equation

$$4x^2 v'' + (4\beta^2 x^2 + 1)v = 0. \quad (291)$$

If we now multiply Equation (290) by $v(x)$ and Equation (291) by $u(x)$ and subtract, we obtain

$$4x^2(vu'' - uv'') + 4x^2(\alpha^2 uv - \beta^2 vu) = 0,$$

or

$$uv'' - vu'' = uv(\alpha^2 - \beta^2), \quad (292)$$

and integration of this equation between the limits of 0 and x gives

$$\begin{aligned} \int_0^x (uv'' - vu'') dx &= (uv' - vu') \Big|_0^x - \int_0^x u'v' dx + \int_0^x v'u' dx \\ &= (\alpha^2 - \beta^2) \int_0^x uv dx, \end{aligned}$$

or

$$\begin{aligned} (\alpha^2 - \beta^2) \int_0^x u(\zeta)v(\zeta) d\zeta &= [u(x)v'(x) - v(x)u'(x)] \\ &\quad - [u(0)v'(0) - v(0)u'(0)]. \end{aligned}$$

Recalling the definitions of u and v we see that

$$u(0) = 0 \cdot J_0(0) = 0,$$

$$v(0) = 0 \cdot J_0(0) = 0,$$

and hence that the above equation becomes

$$\begin{aligned}
 (\alpha^2 - \beta^2) \int_0^x \zeta J_0(\alpha\zeta) J_0(\beta\zeta) d\zeta &= x J_0(\alpha x) J'_0(\beta x) + \frac{1}{2} J_0(\alpha x) J_0(\beta x) \\
 &\quad - x J_0(\beta x) J'_0(\alpha x) - \frac{1}{2} J_0(\beta x) J_0(\alpha x) \\
 &= x [J_0(\alpha x) J'_0(\beta x) - J_0(\beta x) J'_0(\alpha x)]. \quad (293)
 \end{aligned}$$

But from Equation (286) with $n = 0$,

$$\frac{d}{dx} J_0(x) = J_{-1}(x),$$

and from Equation (285)

$$J_{-1}(x) = -J_1(x).$$

Also

$$\frac{d}{dx} J_0(\alpha x) = -\alpha J_1(\alpha x),$$

$$\frac{d}{dx} J_0(\beta x) = -\beta J_1(\beta x).$$

Utilizing this result in Equation (293), we obtain

$$(\alpha^2 - \beta^2) \int_0^x \zeta J_0(\alpha\zeta) J_0(\beta\zeta) d\zeta = x [\alpha J_0(\beta x) J_1(\alpha x) - \beta J_0(\alpha x) J_1(\beta x)]. \quad (294)$$

If we now let $x = c$, $\alpha = \lambda/c$, $\beta = \mu/c$, Equation (294) becomes

$$(\lambda^2 - \mu^2) \int_0^c \zeta J_0\left(\lambda \frac{\zeta}{c}\right) J_0\left(\mu \frac{\zeta}{c}\right) d\zeta = c^3 \left[\frac{\lambda}{c} J_0(\mu) J_1(\lambda) - \frac{\mu}{c} J_0(\lambda) J_1(\mu) \right],$$

and if λ and μ are two *distinct* roots of $J_0(x)$,

$$\int_0^c x J_0\left(\lambda \frac{x}{c}\right) J_0\left(\mu \frac{x}{c}\right) dx = 0 \quad \lambda \neq \mu. \quad (295)$$

This proves the first orthogonality condition. If $\lambda = \mu$ are the *same* zero of $J_0(x) = 0$, we cannot divide the equation, as we did above, by $\lambda^2 - \mu^2$. But by differentiation of Equation (294) with respect to α we have

$$\begin{aligned}
 2\alpha \int_0^x \zeta J_0(\alpha\zeta) J_0(\beta\zeta) d\zeta + (\alpha^2 - \beta^2) \int_0^x \zeta \frac{d}{d\alpha} J_0(\alpha\zeta) J_0(\beta\zeta) d\zeta \\
 = x \left[J_0(\beta x) J_1(\alpha x) + \alpha J_0(\beta x) \frac{d}{d\alpha} J_1(\alpha x) - \beta \frac{d}{d\alpha} J_0(\alpha x) J_1(\beta x) \right]. \quad (296)
 \end{aligned}$$

If we let $x = c$ and $\alpha = \beta = \lambda/c$, where λ is a root of $J_0(x) = 0$, we see that $\alpha^2 - \beta^2 = 0$ and hence the second integral on the left of Equation (296) vanishes. We also notice that

$$J_0(\beta c) = J_0(\lambda) = 0$$

and that by Equations (285) and (286)

$$\frac{d}{d\alpha} J_0(\alpha c) = c J_{-1}(\alpha c) = -c J_1(\alpha c) = -c J_1(\lambda).$$

Hence

$$2 \frac{\lambda}{c} \int_0^c x J_0^2\left(\lambda \frac{x}{c}\right) dx = c \left[-\frac{\lambda}{c} (-c) J_1(\lambda) J_1(\lambda) \right]$$

or, since $\lambda \neq 0$,

$$\int_0^c x J_0^2\left(\lambda \frac{x}{c}\right) dx = \frac{c^2}{2} [J_1^2(\lambda)], \quad (297)$$

which complete the orthogonality proof.

The roots (or zeros) of the function $J_0(x)$ are often needed in the applications; the first twenty are tabulated in Table II (see also Figure 79).

TABLE II
ZEROS OF THE BESSEL FUNCTION $J_0(x)$

$\lambda_1 = 2.405$	$\lambda_{11} = 33.776$
$\lambda_2 = 5.520$	$\lambda_{12} = 36.917$
$\lambda_3 = 8.654$	$\lambda_{13} = 40.058$
$\lambda_4 = 11.792$	$\lambda_{14} = 43.200$
$\lambda_5 = 14.931$	$\lambda_{15} = 46.341$
$\lambda_6 = 18.071$	$\lambda_{16} = 49.483$
$\lambda_7 = 21.212$	$\lambda_{17} = 52.624$
$\lambda_8 = 24.352$	$\lambda_{18} = 55.766$
$\lambda_9 = 27.494$	$\lambda_{19} = 58.907$
$\lambda_{10} = 30.635$	$\lambda_{20} = 62.048$

84. Completion of the solution of Section 73

We considered in Section 73 the problem of finding the temperature in a thin disk insulated on the lateral surfaces and with the boundary conditions

$$\begin{aligned} 1. \quad u(a, t) &= 0, \\ 2. \quad u(r, 0) &= f(r). \end{aligned} \quad (298)$$

The equation satisfied by temperature $u(r, \theta)$ was found to be

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{1}{k} \frac{\partial u}{\partial t} = 0, \quad (299)$$

and assuming a solution of the form

$$u(r, t) = R(r)T(t),$$

we arrived at the two ordinary differential equations

$$T' + k\beta^2 T = 0, \quad (300a)$$

$$R'' + \frac{1}{r}R' + \beta^2 R = 0. \quad (300b)$$

By means of the substitution $s = \beta r$, the second equation reduces to the standard form of Bessel's equation of order zero :

$$s^2 \frac{d^2 R}{ds^2} + s \frac{dR}{ds} + s^2 R = 0,$$

whose solutions are $J_0(s)$ and $Y_0(s)$. Hence the solutions of Equation (300b) are $J_0(r\beta)$ and $Y_0(r\beta)$, and the function

$$u(r, t) = e^{-k\beta^2 t} [bJ_0(r\beta) + cY_0(r\beta)] \quad (301)$$

satisfies Equation (299) whatever the values of the constants β , b , c . But referring to Figure 80, $\lim_{x \rightarrow 0} Y_0(x) = -\infty$, and since $u(0, t)$ is certainly finite, we must eliminate the function Y_0 from our solution. Hence

$$u(r, t) = be^{-k\beta^2 t} J_0(r\beta).$$

From the first boundary condition of Equation (298),

$$u(a, t) = be^{-k\beta^2 t} J_0(a\beta) = 0.$$

This implies $J_0(a\beta) = 0$ and shows that $a\beta$ must be a root of $J_0(x) = 0$,

$$a\beta = \lambda_i, \quad \beta = \lambda_i/a, \quad i = 1, 2, \dots$$

Due to the linearity of our equation, we conclude that the function

$$u(r, t) = \sum_{n=1}^{\infty} b_n e^{-k(\lambda_n/a)^2 t} J_0\left(\lambda_n \frac{r}{a}\right),$$

where the b_n are arbitrary constants, is a solution of Equation (299) which satisfies the first of the boundary conditions of Equation (298). If $u(r, t)$ is to equal $f(r)$ when $t = 0$, we must have, moreover,

$$f(r) = \sum_{n=1}^{\infty} b_n J_0\left(\lambda_n \frac{r}{a}\right). \quad (302)$$

We now obtain the b_n by means of the orthogonality properties of the Bessel functions. Multiplying Equation (302) by

$$r J_0\left(\lambda_m \frac{r}{a}\right)$$

and integrating from 0 to a with respect to r ,

$$\begin{aligned} \int_0^a r f(r) J_0\left(\lambda_m \frac{r}{a}\right) dr &= \sum_{n=1}^{\infty} b_n \int_0^a r J_0\left(\lambda_n \frac{r}{a}\right) J_0\left(\lambda_m \frac{r}{a}\right) dr \\ &= b_m \frac{a^2}{2} [J_1^2(\lambda_m)], \end{aligned}$$

from which,

$$b_m = \frac{2}{a^2 J_1^2(\lambda_m)} \int_0^a r f(r) J_0\left(\lambda_m \frac{r}{a}\right) dr.$$

The solution to the temperature problem thus becomes

$$u(r, t) = \frac{2}{a^2} \sum_{n=1}^{\infty} \frac{1}{J_1^2(\lambda_n)} \left[\int_0^a s f(s) J_0\left(\lambda_n \frac{s}{a}\right) ds \right] J_0\left(\lambda_n \frac{r}{a}\right) e^{-k(\lambda_n/a)^2 t}. \quad (303)$$

Infinite series of the type appearing in Equation (303) are often spoken of as "Fourier-Bessel" series in analogy with ordinary Fourier series.

As a numerical example, assume $f(r) = 1$. Then

$$b_n = \frac{2}{a^2 J_1^2(\lambda_n)} \int_0^a r f(r) J_0\left(\lambda_n \frac{r}{a}\right) dr = \frac{2}{a^2 J_1^2(\lambda_n)} \int_0^a r J_0\left(\lambda_n \frac{r}{a}\right) dr.$$

If we make the change of dummy variable of integration,

$$\begin{aligned} z &= \lambda_n \frac{r}{a} \\ b_n &= \frac{2}{J_1^2(\lambda_n)} \frac{1}{\lambda_n^2} \int_0^{\lambda_n} z J_0(z) dz = \frac{2}{\lambda_n J_1(\lambda_n)}. \end{aligned}$$

[See Eq. (286)]. The numerical result is therefore

$$u(r,t) = 2 \sum_{n=1}^{\infty} \frac{1}{\lambda_n J_1(\lambda_n)} e^{-k(\lambda_n/a)^2 t} J_0\left(\frac{r}{a} \lambda_n\right).$$

Values of the Bessel functions are tabulated, for example, in [30].

85. Solution of Mathieu's equation

The ordinary differential equation

$$y'' + (\lambda - 2q \cos 2x)y = 0 \quad (304)$$

was called a Mathieu equation (see Section 74). If we let

$$x = jz$$

in Equation (304), it reduces to

$$y'' - (\lambda - 2q \cosh 2z)y = 0, \quad (305)$$

which is another form of Mathieu's equation. It will therefore suffice to study Equation (304). While Mathieu's equation may appear simpler than Legendre's or Bessel's equation, we note that it depends on *two* parameters, λ and q . Hence certain features not encountered in these earlier equations will be present in its solution.

We note first that if $q = 0$, Equation (304) reduces to

$$y'' + \lambda y = 0,$$

so that with $\lambda = m^2$, $\cos mx$ and $\sin mx$ are two linearly independent solutions. Hence we should like any solution we obtain of Equation (304) to reduce to $\cos mx$ or $\sin mx$ when $q = 0$. We shall search for *periodic* solutions $y(x, \lambda, q)$ (of period π or 2π) of Mathieu's equation. If $y(x, \lambda, q)$ is one such solution, λ and q must be interrelated, say

$$\lambda = m^2 + \alpha_1 q + \alpha_2 q^2 + \alpha_3 q^3 + \dots \quad (306)$$

Since we want λ to equal m^2 when $q = 0$ we have chosen α_0 as m^2 . Solutions of the form

$$y(x, \lambda, q) = \cos mx + qa_1(x) + q^2 a_2(x) + q^3 a_3(x) + \dots$$

reduce to $\cos mx$ for $q = 0$. These solutions are generally denoted by

$$ce_m(x, q),$$

where "ce" is read "cosine-elliptic." Similarly, solutions of the form

$$\sin mx + qb_1(x) + q^2b_2(x) + q^3b_3(x) + \dots$$

reduce to $\sin mx$ when $q = 0$, and are generally denoted by

$$se_m(x, q),$$

where the "se" is an abbreviation for "sine-elliptic." Both $ce_n(x, q)$ and $se_n(x, q)$ are referred to as Mathieu functions of the first kind of order n .^{*} The first few cosine- and sine-elliptic functions are listed below.

$$\begin{aligned} ce_0(x, q) &= 1 - \frac{1}{2}q \cos 2x + \frac{1}{32}q^2 \cos 4x \\ &\quad - \frac{1}{128}q^3 \left(\frac{1}{9} \cos 6x - 7 \cos 2x \right) + \dots, \\ ce_1(x, q) &= \cos x - \frac{1}{8}q \cos 3x + \frac{1}{64}q^2 \left(\frac{1}{3} \cos 5x - \cos 3x \right) + \dots, \\ ce_2(x, q) &= \cos 2x - \frac{1}{8}q \left(\frac{2}{3} \cos 4x - 2 \right) + \frac{1}{384}q^2 \cos 6x - \dots, \end{aligned} \tag{307}$$

^{*} Since the value of λ is different for ce_n and se_n , these functions cannot be solutions of the *same* Mathieu equation (except in the case $q = 0$). The other independent solutions of Mathieu's equation, $fe_n(x, q)$ and $ge_n(x, q)$ corresponding to $ce_n(x, q)$ and $se_n(x, q)$, respectively, are nonperiodic. The general solution of Equation (304) can therefore be written either as

$$y(x) = A ce_n(x, q) + B fe_n(x, q)$$

or

$$y(x) = C se_n(x, q) + D ge_n(x, q),$$

where A , B , C , and D are arbitrary constants, and n is an integer. (For details of these other solutions see [20].) It will be seen that as in the case of Legendre and Bessel functions of the second kind, the fe_n and ge_n functions do not often appear in the solution of elementary boundary value problems. In particular we shall not need them in our problems.

$$se_1(x, q) = \sin x - \frac{1}{8}q \sin 3x + \frac{1}{64}q^2 \left(\frac{1}{3} \sin 5x + \sin 3x \right) - \dots,$$

$$se_n(x, q) = \sin 2x - \frac{1}{12}q \sin 4x + \frac{1}{384}q^2 \sin 6x - \dots$$

The functions $ce_n(x, q)$ and $se_n(x, q)$ are periodic of period 2π if n is odd and periodic of period π if n is even. We may therefore write

$$\begin{aligned} ce_{2n}(x, q) &= \sum_{\alpha=0}^{\infty} A_{2\alpha}^{(2n)} \cos 2\alpha x, \\ ce_{2n+1}(x, q) &= \sum_{\alpha=0}^{\infty} A_{2\alpha+1}^{(2n+1)} \cos (2\alpha + 1)x, \\ se_{2n}(x, q) &= \sum_{\beta=0}^{\infty} B_{2\beta}^{(2n)} \sin 2\beta x, \\ se_{2n+1}(x, q) &= \sum_{\beta=0}^{\infty} B_{2\beta+1}^{(2n+1)} \sin (2\beta + 1)x, \end{aligned} \tag{308}$$

where the A 's and B 's are known constants depending on the parameter q .

We have seen that Equation (304) can be transformed into Equation (305) by replacing x by jx . Hence

$$ce_n(jx, q) \quad \text{and} \quad se_n(jx, q)$$

are solutions of Equation (305). The solutions of Equation (305) are generally written

$$Ce_n(x, q) \quad \text{and} \quad Se_n(x, q),$$

where

$$\begin{aligned} Ce_n(x, q) &\equiv ce_n(jx, q), \\ Se_n(x, q) &\equiv -j se_n(jx, q), \end{aligned} \tag{309}$$

and are called *modified Mathieu functions* of the first kind of integral order.

86. Orthogonality of the Mathieu functions

We shall now show that the functions $\{ce_n(x, q), se_m(x, q)\}$ are orthogonal in the interval $(0, 2\pi)$. Let

$$y_1(x) = ce_m(x, q) \quad \text{and} \quad y_2(x) = ce_n(x, q)$$

be solutions of the Mathieu equations

$$y'' + (\lambda_1 - 2q \cos 2x)y = 0 \quad (310)$$

and

$$y'' + (\lambda_2 - 2q \cos 2x)y = 0, \quad (311)$$

respectively, with different λ 's but the same q . Multiply Equation (310) by y_2 and Equation (311) by y_1 and subtract. This yields

$$y_2 y_1'' - y_1 y_2'' = (\lambda_2 - \lambda_1) y_1 y_2.$$

Integrate this equation from 0 to ζ :

$$\int_0^\zeta [y_2(x) y_1''(x) - y_1(x) y_2''(x)] dx = (\lambda_2 - \lambda_1) \int_0^\zeta y_1(x) y_2(x) dx.$$

If we integrate by parts as was done in the corresponding development for the Legendre and Bessel functions, we obtain

$$[y_1'(x) y_2(x) - y_2'(x) y_1(x)] \Big|_0^\zeta = (\lambda_2 - \lambda_1) \int_0^\zeta y_1(x) y_2(x) dx. \quad (312)$$

Since $y_1 = ce_m(x, q)$ and $y_2 = ce_n(x, q)$ are both periodic of period π or 2π depending on whether n and m are odd or even, the left-hand side of Equations (312) vanishes for $\zeta = 2\pi$, and

$$(\lambda_2 - \lambda_1) \int_0^{2\pi} ce_m(x, q) ce_n(x, q) dx = 0.$$

If $m \neq n$, $\lambda_1 \neq \lambda_2$ and

$$\int_0^{2\pi} ce_m(x, q) ce_n(x, q) dx = 0, \quad m \neq n.$$

If, instead $n = m$ and n is odd, say $n = 2r + 1$,

$$\begin{aligned} \int_0^{2\pi} ce_{2r+1}^2(x, q) dx &= \int_0^{2\pi} \left[\sum_{\alpha=0}^{\infty} A_{2\alpha+1}^{(2r+1)} \cos(2\alpha+1)x \right]^2 dx \\ &= \pi \sum_{\alpha=0}^{\infty} [A_{2\alpha+1}^{(2r+1)}]^2 \neq 0 \end{aligned}$$

by the orthogonality of the cosine functions. Similarly, if n is even, say $n = 2s$,

$$\int_0^{2\pi} \text{ce}_{2s}^2(x, q) dx = 2\pi[A_0^{(2s)}]^2 + \pi \sum_{\alpha=0}^{\infty} [A_{2\alpha}^{(2s)}]^2 \neq 0.$$

Using the same technique, we can readily show that

$$\int_0^{2\pi} \text{se}_m(x, q) \text{se}_n(x, q) dx = 0, \quad n \neq m,$$

$$\int_0^{2\pi} \text{se}_{2s}^2(x, q) dx = \pi \sum_{\beta=0}^{\infty} [B_{2\beta}^{(2s)}]^2 \neq 0,$$

$$\int_0^{2\pi} \text{se}_{2r+1}^2(x, q) dx = \pi \sum_{\beta=0}^{\infty} [B_{2\beta+1}^{(2r+1)}]^2 \neq 0,$$

and

$$\int_0^{2\pi} \text{ce}_m(x, q) \text{se}_n(x, q) dx = 0.$$

Note that by virtue of Equation (308), this last equation holds whether the q appearing in the cosine-elliptic term is the same as or different from the q appearing in the sine-elliptic term.

87. Completion of the solution of Section 74

In Section 74 we considered (Equation 245) the problem of finding the transverse displacements of an elliptic membrane and saw that a function $u(x, y, t)$ of the form

$$u(x, y, t) = w(x, y) (A \cos \omega t + B \sin \omega t) \quad (313)$$

was a solution of Equation (245) if w satisfied the equation (in elliptic coordinates)

$$\frac{\partial^2 w}{\partial \xi^2} + \frac{\partial^2 w}{\partial \eta^2} + k^2(\cosh 2\xi - \cos 2\eta)w = 0. \quad (314)$$

The membrane was assumed to be an ellipse of semimajor axis α and semiminor axis β , and the focal distance γ (see Equation 247) was so chosen that the membrane became one of the confocal

ellipses, that is $\gamma^2 = \alpha^2 - \beta^2$ (see Equation 251). It was that particular ellipse of the family corresponding to (see Equation 248)

$$\xi = \xi_0 \quad (\text{a constant}),$$

where

$$\tanh \xi_0 = \frac{\beta}{\alpha}, \quad \xi_0 = \arg \tanh \frac{\beta}{\alpha}.$$

The constant k in Equations (249) and (250) was given by

$$k^2 = \frac{\omega^2(\alpha^2 - \beta^2)}{2a^2}. \quad (315)$$

Since the membrane was assumed fixed on the boundary with an initial displacement and zero initial velocity, the boundary conditions, in the elliptic coordinate system ξ, η , become

$$\begin{aligned} 1. \quad & u(\xi_0, \eta, t) = 0, \\ 2. \quad & u(\xi, \eta, 0) = f(\xi, \eta), \\ 3. \quad & \frac{\partial}{\partial t} u(\xi, \eta, 0) = 0. \end{aligned} \quad (316)$$

The solution of Equation (314) led to two Mathieu equations [Equations (252)], whose periodic solutions were found to be

$$Ce_m(\xi, q) \quad \text{or} \quad Se_m(\xi, q)$$

and

$$ce_n(\eta, q) \quad \text{or} \quad se_n(\eta, q),$$

respectively, where

$$2q = k^2. \quad (317)$$

The solutions of Equation (314) must therefore consist of terms of the form

$$Ce_n(\xi, q) ce_n(\eta, q)$$

and

$$Se(\xi, q) se_n(\eta, q),$$

where Ce_n and ce_n correspond to the same value of λ , while Se_n and se_n also correspond to the same value of λ , but not the value appearing in Ce_n and ce_n . Hence the function

$$\begin{aligned} u_n(\xi, \eta, t) = & Ce_n(\xi, q) ce_n(\eta, q) [A_n \cos \omega_n t + B_n \sin \omega_n t] \\ & + Se_n(\xi, q) se_n(\eta, q) [C_n \cos \omega_n t + D_n \sin \omega_n t] \end{aligned}$$

satisfies the partial differential equation of Equation (245). We may use the third boundary condition of Equation (316) to eliminate B_n and D_n . For,

$$\frac{\partial}{\partial t} u_n(\xi, \eta, t) = C_n(\xi, q) ce_n(\eta, q)[- \omega_n A_n \sin \omega_n t + \omega_n B_n \cos \omega_n t] \\ + Se_n(\xi, q) se_n(\eta, q)[- \omega_n C_n \sin \omega_n t + \omega_n D_n \cos \omega_n t]$$

and

$$0 = \frac{\partial}{\partial t} u_n(\xi, \eta, 0) = B_n \omega_n C_n(\xi, q) ce_n(\eta, q) + D_n \omega_n Se_n(\xi, q) se_n(\eta, q).$$

Since the elliptic functions are not identically zero,

$$B_n = 0 = D_n,$$

and we may write

$$u_n(\xi, \eta, t) = A_n C_n(\xi, q) ce_n(\eta, q) \cos \omega_n t + C_n Se_n(\xi, q) se_n(\eta, q) \cos \omega_n t.$$

From the first boundary condition of Equation (316),

$$u_n(\xi_0, \eta, t) = A_n C_n(\xi_0, q) ce_n(\eta, q) \cos \omega_n t \\ + C_n Se_n(\xi_0, q) se_n(\eta, q) \cos \omega_n t = 0.$$

Since A_n and C_n are arbitrary and $ce_n(\eta, q) \neq 0$, $\cos \omega_n t \neq 0$, we must have

$$C_n(\xi_0, q) = 0. \quad (318)$$

It may be shown that corresponding to each n , there is an infinite number of roots, say

$$q_{n1}, q_{n2}, q_{n3}, \dots$$

of Equation (318), and for each q_{nm} we obtain from Equations (315) and (317) a frequency ω_{nm} :

$$\frac{4a^2 q_{nm}}{\alpha^2 - \beta^2} = \omega_{nm}^2. \quad (319)$$

Similarly we obtain from

$$Se_n(\xi_0, q) = 0 \quad (320)$$

an infinite number of roots, say

$$\bar{q}_{n1}, \bar{q}_{n2}, \bar{q}_{n3}, \dots$$

and the corresponding $\bar{\omega}_{nm}$:

$$\bar{\omega}_{nm}^2 = \frac{4a^2 \bar{q}_{nm}}{\alpha^2 - \beta^2}. \quad (321)$$

We therefore see that the linear combination

$$\begin{aligned} u(\xi, \eta, t) = & \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} A_{nm} Ce_n(\xi, q_{nm}) ce_n(\eta, q_{nm}) \cos \omega_{nm} t \\ & + \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} C_{nm} Se_n(\xi, \bar{q}_{nm}) se_n(\eta, \bar{q}_{nm}) \cos \bar{\omega}_{nm} t \end{aligned} \quad (322)$$

satisfies the differential equation and the first and third boundary conditions of Equation (316). It will also satisfy

$$u(\xi, \eta, 0) = f(\xi, \eta)$$

if the A_{nm} and C_{nm} are so chosen that

$$\begin{aligned} u(\xi, \eta, 0) = f(\xi, \eta) = & \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} [A_{nm} Ce_n(\xi, q_{nm}) ce_n(\eta, q_{nm}) \\ & + C_{nm} Se_n(\xi, \bar{q}_{nm}) se_n(\eta, \bar{q}_{nm})]. \end{aligned} \quad (323)$$

The orthogonality conditions proved in Section 86 are useful for expanding an arbitrary function in terms of sine-elliptic and cosine-elliptic functions. However, these orthogonality conditions were based on the assumption that the se_n and ce_n functions corresponded to the same value of q . Clearly the functions of Equation (323) have different values of q . Hence, in order to evaluate the A_{nm} and C_{nm} we must prove another type of orthogonality condition involving Mathieu functions.

We have seen that

$$Ce_n(\xi, q_{nm}) ce_n(\eta, q_{nm}) \quad (324a)$$

and

$$Se_n(\xi, \bar{q}_{nm}) se_n(\eta, \bar{q}_{nm}) \quad (324b)$$

are solutions of the equation

$$\frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} + 2q_{nm}(\cosh 2\xi - \cos 2\eta)v = 0.$$

Hence, calling $v_{nm}(\xi, \eta)$ and $v_{rs}(\xi, \eta)$ two of the functions of Equation (324),

$$\frac{\partial^2}{\partial \xi^2} v_{nm} + \frac{\partial^2}{\partial \eta^2} v_{nm} + 2q_{nm}(\cosh 2\xi - \cos 2\eta)v_{nm} = 0, \quad (325)$$

$$\frac{\partial^2}{\partial \xi^2} v_{rs} + \frac{\partial^2}{\partial \eta^2} v_{rs} + 2q_{rs}(\cosh 2\xi - \cos 2\eta)v_{rs} = 0. \quad (326)$$

Multiplying Equation (325) by v_{rs} and Equation (326) by v_{nm} and subtracting, we obtain

$$\begin{aligned} v_{rs} \frac{\partial^2}{\partial \xi^2} v_{nm} - v_{nm} \frac{\partial^2}{\partial \xi^2} v_{rs} + v_{rs} \frac{\partial^2}{\partial \eta^2} v_{nm} - v_{nm} \frac{\partial^2}{\partial \eta^2} v_{rs} \\ + 2(q_{nm} - q_{rs})(\cosh 2\xi - \cos 2\eta)v_{nm}v_{rs} = 0, \end{aligned}$$

or

$$\begin{aligned} \frac{\partial}{\partial \xi} \left(v_{rs} \frac{\partial}{\partial \xi} v_{nm} - v_{nm} \frac{\partial}{\partial \xi} v_{rs} \right) + \frac{\partial}{\partial \eta} \left(v_{rs} \frac{\partial}{\partial \eta} v_{nm} - v_{nm} \frac{\partial}{\partial \eta} v_{rs} \right) \\ + 2(q_{nm} - q_{rs})(\cosh 2\xi - \cos 2\eta)v_{nm}v_{rs} = 0. \end{aligned}$$

Now integrate both sides of the above equation from 0 to 2π with respect to η and from 0 to ξ_0 with respect to ξ .

$$\begin{aligned} \int_0^{2\pi} \left(v_{rs} \frac{\partial}{\partial \xi} v_{nm} - v_{nm} \frac{\partial}{\partial \xi} v_{rs} \right) \Big|_0^{\xi_0} d\eta + \int_0^{\xi_0} \left(v_{rs} \frac{\partial}{\partial \eta} v_{nm} - v_{nm} \frac{\partial}{\partial \eta} v_{rs} \right) \Big|_0^{2\pi} d\xi \\ + 2(q_{nm} - q_{rs}) \int_0^{2\pi} \int_0^{\xi_0} v_{nm}v_{rs}(\cosh 2\xi - \cos 2\eta) d\xi d\eta = 0. \end{aligned}$$

The first integral on the left vanishes at $\xi = 0$ (see Equation 308 and Section 86); the integrand vanishes at $\xi = \xi_0$ since the membrane was assumed clamped at $\xi = \xi_0$; and the integrand of the second integral vanishes since v_{rs} and v_{nm} are periodic of period 2π in η . Hence we conclude that if $q_{nm} \neq q_{rs}$,

$$\int_0^{2\pi} \int_0^{\xi_0} v_{nm}v_{rs}(\cosh 2\xi - \cos 2\eta) d\xi d\eta = 0. \quad (327)$$

If $n = r$, $m = s$, $q_{nm} = q_{rs}$, and

$$\int_0^{2\pi} \int_0^{\xi_0} v_{nm}^2(\xi, \eta) (\cosh 2\xi - \cos 2\eta) d\xi d\eta \neq 0, \quad (328)$$

since $v_{nm}^2 > 0$, and for any ξ and any η ,

$$\cosh 2\xi - \cos 2\eta \geq 0.$$

By virtue of the orthogonality conditions of Equations (327) and (328), multiplying both sides of Equation (323) by

$$Ce_r(\xi, q_{rs}) ce_r(\eta, q_{rs}) (\cosh 2\xi - \cos 2\eta)$$

and integrating from 0 to ξ_0 with respect to ξ and from 0 to 2π with respect to η we obtain

$$\begin{aligned} \int_0^{\xi_0} \int_0^{2\pi} f(\xi, \eta) Ce_r(\xi, q_{rs}) ce_r(\eta, q_{rs}) (\cosh 2\xi - \cos 2\eta) d\xi d\eta \\ = A_{rs} \int_0^{\xi_0} \int_0^{2\pi} Ce_r^2(\xi, q_{rs}) ce_r^2(\eta, q_{rs}) (\cosh 2\xi - \cos 2\eta) d\xi d\eta. \end{aligned}$$

Hence

$$A_{rs} = \frac{\int_0^{\xi_0} \int_0^{2\pi} f(\xi, \eta) Ce_r(\xi, q_{rs}) ce_r(\eta, q_{rs}) (\cosh 2\xi - \cos 2\eta) d\xi d\eta}{\int_0^{\xi_0} \int_0^{2\pi} Ce_r^2(\xi, q_{rs}) ce_r^2(\eta, q_{rs}) (\cosh 2\xi - \cos 2\eta) d\xi d\eta}.$$

Similarly,

$$C_{rs} = \frac{\int_0^{\xi_0} \int_0^{2\pi} f(\xi, \eta) Se_r(\xi, \bar{q}_{rs}) se_r(\eta, \bar{q}_{rs}) (\cosh 2\xi - \cos 2\eta) d\xi d\eta}{\int_0^{\xi_0} \int_0^{2\pi} Se_r^2(\xi, \bar{q}_{rs}) se_r^2(\eta, \bar{q}_{rs}) (\cosh 2\xi - \cos 2\eta) d\xi d\eta}.$$

We have thus determined the A_{nm} and C_{nm} of Equation (322) and completed the solution of the boundary value problem of Section 74.

88. A unifying principle

We have considered in this and earlier chapters four sets of orthogonal functions: the trigonometric functions, the Legendre polynomials, Bessel functions, and Mathieu functions. All these functions were solutions of second-order linear ordinary differential equations with certain appropriate boundary conditions. It may be reasonable to ask whether there is any general theory of such equations of which the above four examples are merely special

cases. The answer is yes, and is found in the general *Sturm-Liouville theory* of ordinary differential equations (see [12]). This theory considers a differential system of the form

$$\begin{aligned} \frac{d}{dx} \left[p(x) \frac{d}{dx} u(x) \right] + [q(x) + \lambda r(x)] u(x) &= 0, \\ \alpha u(a) + \beta u'(a) &= 0, \\ \gamma u(b) + \delta u'(b) &= 0, \end{aligned} \tag{329}$$

defined over the interval (a, b) , where λ is a parameter. (Actually the theory embraces a slightly more general differential system than the one considered above.)

Any second-order linear differential equation with a parameter, namely,

$$f(x)u''(x) + g(x)u'(x) + [h(x) + \lambda k(x)]u(x) = 0$$

may be written in the form of Equation (329) on multiplying by the integrating factor

$$\frac{1}{f(x)} e^{\int \frac{g(x)}{f(x)} dx}.$$

For then this equation may be written

$$\frac{d}{dx} \left[e^{\int \frac{g}{f} dx} u'(x) \right] + e^{\int \frac{g}{f} dx} \frac{h(x)}{f(x)} u(x) + \lambda e^{\int \frac{g}{f} dx} \frac{k(x)}{f(x)} u(x) = 0,$$

and on letting

$$\begin{aligned} p(x) &= e^{\int \frac{g}{f} dx}, \\ q(x) &= \frac{h(x)}{f(x)} e^{\int \frac{g}{f} dx}, \\ r(x) &= \frac{k(x)}{f(x)} e^{\int \frac{g}{f} dx}, \end{aligned}$$

this becomes Equation (329).

In general, the completely homogeneous system of Equation (329) will not have a solution unidentically zero. However, for certain values of the parameter λ the system may have solutions which are not identically zero. The values of λ for which there

exist such solutions are called *characteristic numbers*. The functions $\phi(x)$ which correspond to these values of λ are called *characteristic functions*. If $\phi(x)$ is a solution of Equations (329) for some λ , then so is $A\phi(x)$ where A is an arbitrary constant. In general, a suitable normalization is applied to the characteristic functions which then specifies A .

For example, the particular differential system

$$\begin{aligned} u''(x) + \lambda u(x) &= 0, \\ u(0) &= 0, \quad u(\pi) = 0, \end{aligned} \tag{330}$$

over the interval $(0, \pi)$ has, in general, no solutions which are not identically zero. However, for

$$\lambda_n = n^2$$

where n is an integer, Equations (330) *do* have a solution unidentically zero, namely,

$$\sin nx.$$

Hence the set of squared integers n^2 are characteristic numbers for the differential system of Equation (330) and the $\{\sin nx\}$ functions are the corresponding characteristic functions.

The fundamental theorem associated with Equations (329) may be stated as follows :

THEOREM. Let $p(x)$, $p'(x)$, $q(x)$, $r(x)$ be continuous functions in the interval $a \leq x \leq b$. Let $p(x) > 0$ in $[a, b]$. Then the Sturm-Liouville system

$$\begin{aligned} [p(x)u']' + [q(x) + \lambda r(x)]u(x) &= 0, \\ \alpha u(a) + \beta u'(a) &= 0, \\ \gamma u(b) + \delta u'(b) &= 0, \end{aligned}$$

has an infinite number of characteristic numbers,

$$\lambda_1 < \lambda_2 < \lambda_3 < \dots$$

and corresponding characteristic functions $\{\phi_n(x)\}$. Any function $f(x)$ which is continuous on $[a, b]$ and has a derivative everywhere

except at a finite number of points in $[a, b]$ can be expanded in an infinite series of the characteristic functions $\{\phi_n(x)\}$,

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$$

where

$$c_n = \int_a^b r(x) f(x) \phi_n(x) dx,$$

and this series converges uniformly to $f(x)$ in any subinterval of $[a, b]$.

We have seen that the trigonometric functions satisfy the equations $u'' + \lambda_n u = 0$ and that $\lambda_n = n^2$ are the characteristic numbers, $\phi_n(x) = \sin nx$, the characteristic functions. Also, the sine functions are orthogonal in the interval $(0, \pi)$.

Legendre's equation, Equation (240), may be written

$$\frac{d}{dx} \left[(1 - x^2) \frac{du}{dx} \right] + \lambda u(x) = 0.$$

Here $p(x) = 1 - x^2$, $q(x) = 0$, $r(x) = 1$. We have seen that the characteristic numbers λ_n are $n(n+1)$ and that the characteristic functions $P_n(x)$, the Legendre polynomials, are orthogonal with respect to the weighting function $r(x) = 1$.

Also, Bessel's equation of order zero, Equation (243), may be written in the form

$$\frac{d}{dx} [xu'(x)] + \lambda xu(x) = 0.$$

We have seen that for the interval $[0, 1]$ the characteristic numbers λ are λ_n^2 , where the λ_n are the roots of the Bessel function $J_0(x) = 0$. Also, the functions $J_0(\lambda_n x)$ are orthogonal with respect to the weighting function $r(x) = x$.

Actually, a slight modification of the Sturm-Liouville theory has to be used in the case of Legendre and Bessel functions, since the condition $p(x) > 0$ of the theorem is not satisfied in these two cases.

PROBLEMS

1. Derive the expression for the Laplacian

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

in spherical coordinates.

2. Derive the expression for

$$\nabla^2 w + \frac{\omega^2}{a^2} w = 0$$

in elliptic coordinates.

3. Show that the three-dimensional heat equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = a^2 \frac{\partial u}{\partial t}$$

becomes

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = a^2 \frac{\partial u}{\partial t}$$

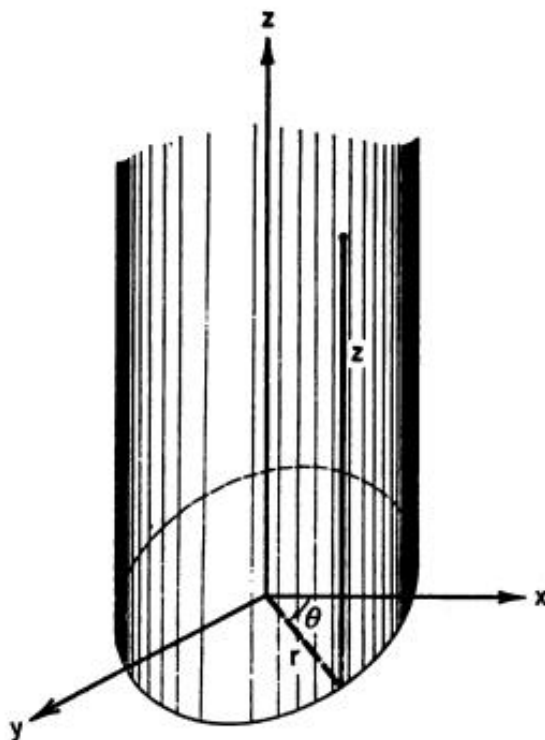
 in cylindrical coordinates (r, θ, z) . See Figure 81.


Fig. 81

4. Determine the constants $\alpha_i, \beta_j, \gamma_k$ so that the functions

$$\begin{aligned} A(x) &= \alpha_1, \\ B(x) &= \beta_1 + \beta_2 x, \\ C(x) &= \gamma_1 + \gamma_2 x + \gamma_3 x^2, \end{aligned}$$

become orthonormal in the interval $(0, 1)$.

5. Show that the functions

$$\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \dots$$

form an orthonormal set in $[-\pi, \pi]$.

6. The *beta function*, defined by the integral

$$B(u, v) = \int_0^1 x^{u-1} (1-x)^{v-1} dx, \quad u, v > 0$$

often appears in mathematical physics. Prove that $B(u, v) = B(v, u)$.

7. Prove that the beta function is related to the gamma function by the equation

$$B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}.$$

[Hint: Make the substitution $1-x = \sin^2 \theta$ in the expression for the beta function and the transformation $x = r \cos^2 \theta, y = r \sin^2 \theta$ in $\Gamma(u)\Gamma(v) = \int_0^\infty \int_0^\infty x^{u-1} e^{-x} y^{v-1} e^{-y} dx dy$.]

8. Prove by the result of Problem 7 that

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

9. Show that

$$\Gamma\left(\frac{2m+1}{2}\right) = \frac{(2m)!}{m! 2^{2m}} \sqrt{\pi}$$

where m is a positive integer.

10. Derive, in detail, the solutions of Legendre's equation

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

for $|x| > 1$ and $|x| < 1$.

11. Show that $P_n(x)$, the Legendre polynomial of order n , is an odd function if n is odd and an even function if n is even.

12. Prove that

$$P_n(1) = 1, \quad n = 0, 1, 2, \dots$$

13. Establish the formulas

$$P_n(0) = \frac{(-1)^{n/2} n!}{2^n \left[\left(\frac{n}{2} \right)! \right]^2} \quad n \text{ even,}$$

$$P_n(0) = 0 \quad n \text{ odd.}$$

14. Show that

$$P_n^{(r)}(1) = \frac{(n+r)!}{2^r r! (n-r)!}$$

$$P_n^{(r)}(-1) = (-1)^{n+r} \frac{(n+r)!}{2^r r! (n-r)!}.$$

15. Establish *Rodrigues' formula*

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

[Hint : Integrate, n times, the expression for $P_n(x)$ (Equation 268) from 0 to x .]

16. Show that the Legendre polynomials satisfy the recursion formula

$$nP_{n-1}(x) - (2n+1)xP_n(x) + (n+1)P_{n+1}(x) = 0.$$

[Hint : Consider the generating function of Equation (275) for the Legendre polynomials and differentiate it with respect to λ .]

17. Establish the recursion formula

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x).$$

[Hint : Differentiate the generating function of Equation (275) with respect to x .]

18. Prove that

$$\int_x^1 P_n(x) dx = \frac{1}{2n+1} [P_{n-1}(x) - P_{n+1}(x)].$$

19. Establish formulas for

$$(a) \quad \int_0^1 x P_n(x) dx,$$

$$(b) \quad \int_0^1 x^m P_0(x) dx,$$

$$(c) \quad \int_0^1 x^m P_1(x) dx.$$

20. Generalize the results of Problem 19 to

$$\int_0^1 x^m P_n(x) dx.$$

21. Prove

$$\int_{-1}^1 f(x) P_n(x) dx = \frac{(-1)^n}{2^n n!} \int_{-1}^1 (x^2 - 1)^n \frac{d^n f(x)}{dx^n} dx.$$

[Hint : Use Rodrigues' formula.]

22. Expand the function $f(x) = x^3$ in a series of Legendre polynomials.

23. Expand the function $f(x)$

$$f(x) = -1 \quad -1 < x < 0,$$

$$f(x) = +1 \quad 0 < x < 1,$$

in a series of Legendre polynomials.

24. Expand the function

$$f(x) = -x \quad -1 < x < 0,$$

$$f(x) = +x \quad 0 < x < 1,$$

in a series of Legendre polynomials.

25. Obtain the Legendre series expansion for the function

$$f(x) = 0 \quad -1 < x < 0,$$

$$f(x) = x \quad 0 < x < 1.$$

26. Let a sphere of unit radius have its lower hemisphere kept at a constant temperature U_0 , and its upper half kept at zero temperature. Find the steady-state temperature distribution $u(r, \theta)$ inside the sphere.

27. Solve Problem 26 for the temperature outside the sphere.

28. A hemisphere has its curved surface held at a fixed electrostatic potential while its bounding plane is kept at zero potential. Find the electrostatic potential at any interior point.

29. A hemisphere has a magnetostatic potential ψ which is symmetrical about the vertical axis through the center of the sphere, while on the bounding plane, $x = 0$, $\partial\psi/\partial\theta = 0$. Find the magnetic potential at any interior point.

30. Derive in detail the infinite series solution of Bessel's equation.

31. Obtain the second linearly independent solution, $Y_n(x)$ of Bessel's equation of integral order n .

32. Prove

$$J_n(-x) = (-1)^n J_n(x), \quad n = 0, 1, 2, \dots$$

33. Establish the formulas

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x,$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x.$$

34. Establish the following recursion formula for the Bessel functions of the first kind.

$$xJ'_n(x) + nJ_n(x) = xJ_{n-1}(x).$$

35. Establish the identity

$$\frac{d}{dx} [x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x).$$

36. Prove the recursion formula

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x).$$

37. Prove that

$$\int_0^x \zeta J_0(\zeta) d\zeta = xJ_1(x).$$

38. Calculate

$$\int_0^\zeta J_4(x) dx.$$

[Hint : Write

$$\int J_4(x) dx = - \int x^3 [-x^{-3} J_4(x)] dx$$

and apply the result of Problem 37.]

39. Evaluate

$$\int_0^1 x^3 J_0(x) dx.$$

[Hint : Use Problem 37 and integrate by parts.]

40. Expand the function

$$f(x) = x \quad 0 < x < 2$$

into a Fourier-Bessel series of Bessel functions of the first kind and order zero.

41. Expand $f(x)$:

$$f(x) = h, \quad 0 < x < 1$$

in a Fourier-Bessel series of $J_0(\lambda_i x)$, where λ_i are the positive roots of $J_0(\zeta) = 0$.

42. A heavy string of length L and weight w lb/ft hangs from a fixed point. If the initial displacement of the string $u(x, 0)$ is a given function of x , say $f(x)$, and the initial velocity is zero, find the displacement $u(x, t)$. [Hint : The equation of the string is

$$x \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} = \frac{1}{g} \frac{\partial^2 u}{\partial t^2},$$

where g is the gravitational constant. See [25], Section 11.4.]

43. Determine the longitudinal vibrations of a bar (Equation 14) fixed at the end point, $x = L$ which is given an initial displacement $f(x)$ and has zero initial velocity. Assume the material of the bar is such that $EA = kx$, $EA = k'$, where k and k' are constants.

44. A thin flat circular disk has its lateral surfaces insulated, while its circumference is held at zero temperature. Find the temperature $u(r, t)$ in the disk if the initial temperature is a function of the radius only, $u(r, 0) = r^2$.

45. Find the temperature in a semi-infinite right circular cylinder whose base is insulated, whose lateral surface is held at zero temperature, and whose initial temperature $u(r, 0) = f(r)$ is independent of θ and z .

46. Find the temperature $u(r, t)$ in the cylinder of Problem 45, if the cylinder has its *lateral* surfaces also insulated.

47. Find the transient temperature distribution $u(r, t)$ in an infinite right circular cylinder whose lateral surfaces are insulated and whose initial temperature is $u(r, 0) = f(r)$.

48. A right circular cylinder of radius a and height h has one base and its lateral surface kept at zero temperature. Find the steady-state temperature $u(r, z)$ in the cylinder if the other base is kept at a temperature $f(r)$ independent of θ .

49. Solve Problem 48 if the lateral surface of the cylinder is insulated instead of being held at zero temperature.

50. A circular membrane, fixed on the boundary, has an initial displacement $u(r, 0) = f(r)$ and a zero initial velocity. Find the motion $u(r, t)$ of the membrane.

51. Find the equation of the transverse vibrations of the membrane of Problem 50 if the membrane has an initial velocity

$$\frac{\partial}{\partial t} u(r, 0) = g(r)$$

and a zero initial displacement.

52. A right circular cylinder of radius a and height h has its entire surface kept at zero temperature. Find the temperature $u(r, z, t)$ at any point and at any time if the initial temperature of the cylinder $u(r, z, 0)$ is a given function, $f(r, z)$.

53. Solve Problem 52 if the lateral surface is insulated, and the other boundary condition remains unchanged.

54. A right semicircular cylinder of radius a and height h has its bases and curved surface kept at zero temperature. Find the temperature $u(r, z, t)$ in the cylinder if the plane surface is insulated. Assume that the initial temperature $u(r, z, 0) = f(r, z)$.

55. A thin circular disk is completely insulated. Find the temperature $u(r, \theta, t)$ in the disk if initially $u(r, \theta, 0) = f(r, \theta)$.

56. A semi-infinite cylinder has its curved surface kept at zero temperature, its base insulated, and has an initial temperature $f(r, \theta)$ independent of z . Find the transient temperature distribution $u(r, \theta, t)$.

57. Solve Problem 56 if the entire surface of the cylinder is *insulated*.

58. An infinite right circular cylinder has its entire surface held at zero temperature. If its initial temperature $u(r, \theta, 0) = f(r, \theta)$, find the temperature $u(r, \theta, t)$ in the cylinder.

59. Find the temperature $u(r, \theta, z)$ in a right circular cylinder of height h if the lateral surface and one base are kept at zero temperature, while the other base, $z = 0$, is kept at a temperature $u(r, \theta, 0) = f(r, \theta)$.

60. A semicircular right cylinder of height h has its curved surfaces insulated, its plane face and one base held at zero temperature. Find the temperature $u(r, \theta, z)$ in the cylinder if the base ($z = 0$) is kept at a given temperature $f(r, \theta)$.

61. A circular membrane clamped on the boundary has an initial displacement $u(r, \theta, 0) = f(r, \theta)$ and a zero initial velocity. Find the transverse vibrations of the membrane.

62. Find $u(r, \theta, t)$ for the membrane of Problem 61 if

$$u(r, \theta, 0) = 0, \quad \frac{\partial}{\partial t} u(r, \theta, 0) = g(r, \theta).$$

63. Show the equivalence of

$$y'' + (\lambda - 2q \cos 2x)y = 0$$

and

$$y'' - (\lambda - 2q \cosh 2x)y = 0$$

under the transformation $x = jz$.

64. Derive the cosine-elliptic periodic solutions

$$ce_0(x, q) \quad \text{and} \quad ce_1(x, q)$$

of Mathieu's equation.

65. Derive the sine-elliptic periodic solutions

$$se_1(x, q) \quad \text{and} \quad se_2(x, q)$$

of Mathieu's equation.

66. Establish, in detail, the orthogonality of the sine-elliptic functions.
67. Consider an elliptic membrane, clamped on its boundary with zero initial displacement and initial velocity a given function of ξ and η . Find the displacement $u(\xi, \eta, t)$ of the membrane.
68. Find the temperature in an infinite right elliptic cylinder whose lateral surfaces are kept at zero temperature and whose initial temperature distribution is independent of the x -axis.
69. An elliptical disk has its lateral surfaces insulated. Find the temperature $u(\xi, \eta, t)$ in the disk if its circumference is kept at zero temperature, and its initial temperature distribution $u(\xi, \eta, 0) = f(\xi, \eta)$.
70. A semi-infinite right elliptical cylinder has its curved surface kept at zero temperature. Find the temperature in the cylinder if its initial rate of temperature change $\partial/\partial t u(\xi, \eta, 0)$ is a given function $g(\xi, \eta)$ and the base is insulated.

Chapter VI

PROPERTIES OF SECOND-ORDER PARTIAL DIFFERENTIAL EQUATIONS

89. Introduction

In keeping with the modern spirit of partial differential equations and the practical approach of this book, we have been engaged so far in the study of "boundary value problems," that is, we have been interested in differential *systems*. However, we can obtain a deeper insight into the structure of partial differential equations by studying some of their general properties. Even though some of our conclusions may be generalized to more complicated equations, we shall consider in this chapter only linear, homogeneous, second-order partial differential equations in two independent variables with constant coefficients, that is, equations of the type

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} + D \frac{\partial u}{\partial x} + E \frac{\partial u}{\partial y} + Fu = 0, \quad (331)$$

where A , B , C , D , E , and F are constants. It will be noted that most of the equations studied in previous chapters belong to this class.

Our first task will consist in classifying equations of the above form into three distinct types depending on whether the quantity $B^2 - 4AC$ is positive, negative, or zero. The procedure followed requires a transformation of the independent variables in Equation (331). We shall then see what simplifications are possible by specializing the transformations, and in this way reduce the equation to a form which is more readily analyzed.

90. Transformation of variables

Consider the most general transformation of the independent variables x and y of Equation (331), namely,

$$\xi = \xi(x, y), \quad \eta = \eta(x, y), \quad (332)$$

and compute the derivatives of u , using ξ and η as intermediate variables :

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x},$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial y},$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u}{\partial \xi^2} \left(\frac{\partial \xi}{\partial x} \right)^2 + 2 \frac{\partial^2 u}{\partial \xi \partial \eta} \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta^2} \left(\frac{\partial \eta}{\partial x} \right)^2 \\ &\quad + \frac{\partial u}{\partial \xi} \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial u}{\partial \eta} \frac{\partial^2 \eta}{\partial x^2}, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial^2 u}{\partial \xi^2} \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial y} + \frac{\partial^2 u}{\partial \xi \partial \eta} \left[\frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x} \right] \\ &\quad + \frac{\partial^2 u}{\partial \eta^2} \frac{\partial \eta}{\partial x} \frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \xi} \frac{\partial^2 \xi}{\partial x \partial y} + \frac{\partial u}{\partial \eta} \frac{\partial^2 \eta}{\partial x \partial y}, \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 u}{\partial y^2} &= \frac{\partial^2 u}{\partial \xi^2} \left(\frac{\partial \xi}{\partial y} \right)^2 + 2 \frac{\partial^2 u}{\partial \xi \partial \eta} \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y} + \frac{\partial^2 u}{\partial \eta^2} \left(\frac{\partial \eta}{\partial y} \right)^2 \\ &\quad + \frac{\partial u}{\partial \xi} \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial u}{\partial \eta} \frac{\partial^2 \eta}{\partial y^2}. \end{aligned}$$

If we substitute these values into Equation (331) we obtain

$$A' \frac{\partial^2 u}{\partial \xi^2} + B' \frac{\partial^2 u}{\partial \xi \partial \eta} + C' \frac{\partial^2 u}{\partial \eta^2} + D' \frac{\partial u}{\partial \xi} + E' \frac{\partial u}{\partial \eta} + F' u = 0, \quad (333)$$

where

$$A' = A \left(\frac{\partial \xi}{\partial x} \right)^2 + B \frac{\partial \xi}{\partial x} \frac{\partial \xi}{\partial y} + C \left(\frac{\partial \xi}{\partial y} \right)^2,$$

$$B' = 2A \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial x} + B \left[\frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x} \right] + 2C \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y},$$

$$C' = A \left(\frac{\partial \eta}{\partial x} \right)^2 + B \frac{\partial \eta}{\partial x} \frac{\partial \eta}{\partial y} + C \left(\frac{\partial \eta}{\partial y} \right)^2,$$

$$\begin{aligned}
 D' &= A \frac{\partial^2 \xi}{\partial x^2} + B \frac{\partial^2 \xi}{\partial x \partial y} + C \frac{\partial^2 \xi}{\partial y^2} + D \frac{\partial \xi}{\partial x} + E \frac{\partial \xi}{\partial y}, \\
 E' &= A \frac{\partial^2 \eta}{\partial x^2} + B \frac{\partial^2 \eta}{\partial x \partial y} + C \frac{\partial^2 \eta}{\partial y^2} + D \frac{\partial \eta}{\partial x} + E \frac{\partial \eta}{\partial y}, \\
 F' &= F.
 \end{aligned}
 \tag{334}$$

Equation (333) is then Equation (331) expressed in terms of the new variables ξ and η .

91. Canonical forms

We shall consider what simplifications can be made in Equation (333) under the assumption that $B^2 - 4AC$ is positive, negative, or zero. Assume for the moment that $B^2 - 4AC > 0$. Then A' and C' may be written

$$\begin{aligned}
 A' &= A \left(\frac{\partial \xi}{\partial x} - \lambda_1 \frac{\partial \xi}{\partial y} \right) \left(\frac{\partial \xi}{\partial x} - \lambda_2 \frac{\partial \xi}{\partial y} \right), \\
 C' &= C \left(\frac{\partial \eta}{\partial x} - \lambda_1 \frac{\partial \eta}{\partial y} \right) \left(\frac{\partial \eta}{\partial x} - \lambda_2 \frac{\partial \eta}{\partial y} \right),
 \end{aligned}$$

where λ_1 and λ_2 are the roots of the quadratic algebraic equation

$$A\lambda^2 + B\lambda + C = 0,$$

and hence are real and unequal.

Suppose we choose $\xi(x, y)$ and $\eta(x, y)$ such that

$$\frac{\partial \xi}{\partial x} - \lambda_1 \frac{\partial \xi}{\partial y} = 0 \quad \text{and} \quad \frac{\partial \eta}{\partial x} - \lambda_2 \frac{\partial \eta}{\partial y} = 0,$$

and the transformation of Equation (332) is reversible, that is, that we can solve for x and y in terms of ξ and η . For example, if we let

$$\xi = \lambda_1 x + y, \quad \eta = \lambda_2 x + y. \tag{335}$$

Then certainly,

$$\frac{\partial \xi}{\partial x} - \lambda_1 \frac{\partial \xi}{\partial y} = 0, \quad \frac{\partial \eta}{\partial x} - \lambda_2 \frac{\partial \eta}{\partial y} = 0,$$

and furthermore we can solve for x and y from Equation (335):

$$x = \frac{\xi - \eta}{\lambda_1 - \lambda_2}, \quad y = \frac{-\lambda_2 \xi + \lambda_1 \eta}{\lambda_1 - \lambda_2},$$

since by hypothesis $\lambda_1 \neq \lambda_2$ and hence $\lambda_1 - \lambda_2 \neq 0$. With such a transformation $A' = 0$, $C' = 0$, and Equation (333) assumes the form

$$B' \frac{\partial^2 u}{\partial \xi \partial \eta} + D' \frac{\partial u}{\partial \xi} + E' \frac{\partial u}{\partial \eta} + F' u = 0. \quad (336)$$

We must check whether B' vanishes, in which case Equation (336) would be a first-order equation. From the quadratic equation $A\lambda^2 + B\lambda + C = 0$ we see that

$$B = -A(\lambda_1 + \lambda_2), \quad C = A\lambda_1\lambda_2.$$

[If A were equal to zero, we could interchange the roles of x and y . If both A and C were zero, Equation (333) would already be in the form of Equation (336).]

The coefficient B' can thus be written

$$\begin{aligned} B' &= A \left[2 \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial x} - (\lambda_1 + \lambda_2) \left(\frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x} \right) + 2\lambda_1\lambda_2 \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y} \right] \\ &= A \left[\frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial x} - \lambda_1 \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x} - \lambda_2 \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} + \lambda_1\lambda_2 \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial x} \right. \\ &\quad \left. - \lambda_2 \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial x} - \lambda_1 \frac{\partial \xi}{\partial x} \frac{\partial \eta}{\partial y} + \lambda_1\lambda_2 \frac{\partial \xi}{\partial y} \frac{\partial \eta}{\partial y} \right] \\ &= A \left[\left(\frac{\partial \xi}{\partial x} - \lambda_1 \frac{\partial \xi}{\partial y} \right) \left(\frac{\partial \eta}{\partial x} - \lambda_2 \frac{\partial \eta}{\partial y} \right) + \left(\frac{\partial \xi}{\partial x} - \lambda_2 \frac{\partial \xi}{\partial y} \right) \left(\frac{\partial \eta}{\partial x} - \lambda_1 \frac{\partial \eta}{\partial y} \right) \right] \\ &= A \left(\frac{\partial \xi}{\partial x} - \lambda_2 \frac{\partial \xi}{\partial y} \right) \left(\frac{\partial \eta}{\partial x} - \lambda_1 \frac{\partial \eta}{\partial y} \right), \end{aligned}$$

since

$$\frac{\partial \xi}{\partial x} - \lambda_1 \frac{\partial \xi}{\partial y} = 0.$$

But

$$\frac{\partial \xi}{\partial x} - \lambda_2 \frac{\partial \xi}{\partial y} \neq 0 \neq \frac{\partial \eta}{\partial x} - \lambda_1 \frac{\partial \eta}{\partial y},$$

for, from Equation (335),

$$\frac{\partial \xi}{\partial x} - \lambda_2 \frac{\partial \xi}{\partial y} = \lambda_1 - \lambda_2 \neq 0,$$

$$\frac{\partial \eta}{\partial x} - \lambda_1 \frac{\partial \eta}{\partial y} = \lambda_2 - \lambda_1 \neq 0.$$

Hence

$$B' \neq 0,$$

and we may divide Equation (336) by B' to obtain

$$\frac{\partial^2 u}{\partial \xi \partial \eta} + d' \frac{\partial u}{\partial \xi} + e' \frac{\partial u}{\partial \eta} + f' u = 0, \quad (337)$$

where

$$d' = \frac{D'}{B'}, \quad e' = \frac{E'}{B'}, \quad f' = \frac{F'}{B'}.$$

Equation (337) is the *canonical form* of Equation (331), corresponding to the condition $B^2 - 4AC > 0$. Any linear differential equation such as Equation (331) in which $B^2 - 4AC > 0$ is said to be of *hyperbolic type*, and can be reduced to the form of Equation (337). We shall examine this equation in more detail in later sections.

Suppose now that $B^2 - 4AC = 0$. Then the roots of the algebraic equation $A\lambda^2 + B\lambda + C = 0$ are real and equal, say $\lambda = \lambda_1 = \lambda_2$. In this case the coefficients A' , B' , C' become

$$A' = A \left(\frac{\partial \xi}{\partial x} - \lambda \frac{\partial \xi}{\partial y} \right)^2,$$

$$B' = 2A \left(\frac{\partial \xi}{\partial x} - \lambda \frac{\partial \xi}{\partial y} \right) \left(\frac{\partial \eta}{\partial x} - \lambda \frac{\partial \eta}{\partial y} \right),$$

$$C' = A \left(\frac{\partial \eta}{\partial x} - \lambda \frac{\partial \eta}{\partial y} \right)^2.$$

If we choose a ξ and an η such that

$$\frac{\partial \xi}{\partial x} - \lambda \frac{\partial \xi}{\partial y} \neq 0, \quad \frac{\partial \eta}{\partial x} - \lambda \frac{\partial \eta}{\partial y} = 1,$$

then

$$B' = C' = 0 \quad \text{and} \quad A' \neq 0.$$

For example, the reversible transformation

$$\eta = \lambda x + y, \quad \xi = x,$$

transforms Equation (333) into the equation

$$A' \frac{\partial^2 u}{\partial \xi^2} + D' \frac{\partial u}{\partial \xi} + E' \frac{\partial u}{\partial \eta} + F' u = 0, \quad (338)$$

and since $A' \neq 0$, we can divide through and obtain

$$\frac{\partial^2 u}{\partial \xi^2} + d' \frac{\partial u}{\partial \xi} + e' \frac{\partial u}{\partial \eta} + f' u = 0, \quad (339)$$

where

$$d' = \frac{D'}{A'}, \quad e' = \frac{E'}{A'}, \quad f' = \frac{F'}{A'}.$$

Any equation of the form of Equation (331) with $B^2 - 4AC = 0$ can be reduced to the canonical form of Equation (339), and is called an equation of *parabolic type*.

Finally, consider the case where $B^2 - 4AC < 0$. The roots λ_1 and λ_2 of $A\lambda^2 + B\lambda + C = 0$ are in this case complex conjugate numbers, say λ and $\bar{\lambda}$. If, as in the hyperbolic equation, we choose ξ and η such that

$$\frac{\partial \xi}{\partial x} - \lambda \frac{\partial \xi}{\partial y} = 0, \quad \frac{\partial \eta}{\partial x} - \bar{\lambda} \frac{\partial \eta}{\partial y} = 0,$$

the constants A' and C' will vanish, and our equation becomes identical with Equation (336). However ξ and η are now complex variables. That is,

$$\begin{aligned} \xi &= \phi(x, y) + j\psi(x, y), \\ \eta &= \phi(x, y) - j\psi(x, y) = \bar{\xi}. \end{aligned}$$

In order to remain in the domain of reals, we shall write our equation in terms of the real variables ϕ and ψ . From the above equations, by addition and subtraction, we have

$$\phi = \frac{1}{2}(\xi + \eta), \quad \psi = \frac{1}{2j}(\xi - \eta).$$

Hence

$$\begin{aligned} \frac{\partial u}{\partial \xi} &= \frac{\partial u}{\partial \phi} \frac{\partial \phi}{\partial \xi} + \frac{\partial u}{\partial \psi} \frac{\partial \psi}{\partial \xi} = \frac{1}{2} \left(\frac{\partial u}{\partial \phi} - j \frac{\partial u}{\partial \psi} \right), \\ \frac{\partial u}{\partial \eta} &= \frac{\partial u}{\partial \phi} \frac{\partial \phi}{\partial \eta} + \frac{\partial u}{\partial \psi} \frac{\partial \psi}{\partial \eta} = \frac{1}{2} \left(\frac{\partial u}{\partial \phi} + j \frac{\partial u}{\partial \psi} \right), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 u}{\partial \xi \partial \eta} &= \frac{1}{2} \left[\frac{\partial^2 u}{\partial \phi^2} \frac{\partial \phi}{\partial \eta} + \frac{\partial^2 u}{\partial \phi \partial \psi} \frac{\partial \psi}{\partial \eta} - j \frac{\partial^2 u}{\partial \psi \partial \phi} \frac{\partial \phi}{\partial \eta} - j \frac{\partial^2 u}{\partial \psi^2} \frac{\partial \psi}{\partial \eta} \right] \\ &= \frac{1}{4} \left(\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial \psi^2} \right). \end{aligned}$$

Introducing this second change of variables, Equation (336) becomes

$$\begin{aligned} \frac{1}{4}B' \left(\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial \psi^2} \right) + \frac{D'}{2} \left(\frac{\partial u}{\partial \phi} - j \frac{\partial u}{\partial \psi} \right) \\ + \frac{E'}{2} \left(\frac{\partial u}{\partial \phi} + j \frac{\partial u}{\partial \psi} \right) + F'u = 0. \end{aligned}$$

Since $B' \neq 0$ by the same arguments used in the hyperbolic case, we can divide by $B'/4$, and the above equation becomes

$$\frac{\partial^2 u}{\partial \phi^2} + \frac{\partial^2 u}{\partial \psi^2} + \alpha' \frac{\partial u}{\partial \phi} + j\beta' \frac{\partial u}{\partial \psi} + \gamma' u = 0, \quad (340)$$

where α' , $j\beta'$, and γ' are real. In fact,

$$\alpha' = \frac{2(D' + E')}{B'}, \quad \beta' = \frac{2(E' - D')}{B'}, \quad \gamma' = 4\frac{F'}{B'}.$$

Any equation of the form of Equation (331) with $B^2 - 4AC < 0$ can be reduced to the canonical form of Equation (340), and is called an equation of *elliptic type*.

92. The hyperbolic equation

We have seen that the general hyperbolic equation can be written

$$\frac{\partial^2 u}{\partial x \partial y} + a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + cu = 0. \quad (341)$$

In order to simplify Equation (341), introduce the new variable w defined by the equation

$$u(x, y) = w(x, y)e^{-bx-ay}.$$

Then

$$\frac{\partial u}{\partial x} = \frac{\partial w}{\partial x} e^{-bx-ay} - bw e^{-bx-ay},$$

$$\frac{\partial u}{\partial y} = \frac{\partial w}{\partial y} e^{-bx-ay} - aw e^{-bx-ay},$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 w}{\partial x \partial y} e^{-bx-ay} - a \frac{\partial w}{\partial x} e^{-bx-ay} - b \frac{\partial w}{\partial y} e^{-bx-ay} + abwe^{-bx-ay}.$$

Substituting these expressions in Equation (341), we obtain the equation

$$\frac{\partial^2 w}{\partial x \partial y} + (c - ab)w = 0.$$

A solution of this equation leads to Bessel's equation (see Problem 13 at the end of this chapter).

93. The wave equation and the Cauchy problem

We have considered at some length in previous sections the wave equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{a^2} \frac{\partial^2 u}{\partial t^2} = 0, \quad (342)$$

which, if we let $y = at$, assumes the form

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0. \quad (343)$$

Since $A = 1$, $B = 0$, $C = -1$, $B^2 - 4AC = 4 > 0$, we conclude that this equation is of hyperbolic type. The roots of the corresponding algebraic equation

$$A\lambda^2 + B\lambda + C = 0$$

are ± 1 ; hence, if we let

$$\xi = x + y, \quad \eta = x - y,$$

Equation (343) reduces to

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0.$$

This equation is readily integrated, its most general solution being

$$u = \phi(\xi) + \psi(\eta),$$

where ϕ and ψ are arbitrary functions. Reverting to the original variables x, y we may also write

$$u = \phi(x + y) + \psi(x - y),$$

and hence

$$u(x, t) = \phi(x + at) + \psi(x - at)$$

is the general solution of the wave equation of Equation (342).

Suppose we wish to solve the wave equation (for example, the vibrating string equation) with the following initial conditions :

$$u(x,0) = f(x), \quad \frac{\partial}{\partial t} u(x,0) = g(x), \quad (344)$$

where $f(x)$ and $g(x)$ are arbitrary functions. This classical problem in the theory of partial differential equations is known as *Cauchy's problem*. With

$$u(x,t) = \phi(x+at) + \psi(x-at)$$

we have, at $t = 0$,

$$u(x,0) = f(x) = \phi(x) + \psi(x).$$

Also

$$\frac{\partial u}{\partial t} = a\phi'(x+at) - a\psi'(x-at)$$

and

$$\frac{\partial}{\partial t} u(x,0) = g(x) = a\phi'(x) - a\psi'(x).$$

Thus the conditions of Equations (344) require that

$$f(x) = \phi(x) + \psi(x), \quad \frac{g(x)}{a} = \phi'(x) - \psi'(x). \quad (345)$$

Integrating the second of the above equations between 0 and x gives

$$\phi(x) - \psi(x) = \frac{1}{a} \int_0^x g(z) dz + K.$$

If we add this to the first of Equations (345) and divide by 2,

$$\phi(x) = \frac{1}{2}f(x) + \frac{1}{2a} \int_0^x g(z) dz + \frac{K}{2}, \quad (346)$$

and if we subtract, and divide by 2,

$$\psi(x) = \frac{1}{2}f(x) - \frac{1}{2a} \int_0^x g(z) dz - \frac{K}{2}. \quad (347)$$

Replace x by $x+at$ in Equation (346) and by $x-at$ in Equation (347) and add; then

$$\begin{aligned} \phi(x+at) + \psi(x-at) &= \frac{1}{2}[f(x+at) + f(x-at)] \\ &\quad + \frac{1}{2a} \int_0^{x+at} g(z) dz + \frac{1}{2a} \int_{x-at}^0 g(z) dz. \end{aligned}$$

But $u(x, t) = \phi(x + at) + \psi(x - at)$. Hence the solution of the Cauchy problem is

$$u(x, t) = \frac{1}{2}[f(x + at) + f(x - at)] + \frac{1}{2a} \int_{x-at}^{x+at} g(x) dx. \quad (348)$$

(See Problem 8 of Chapter III.)

94. The homogeneous equation of hyperbolic type

If, in Equation (331), $D = E = F = 0$, the equation becomes homogeneous* in the second derivatives, namely,

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = 0. \quad (349)$$

If $B^2 - 4AC > 0$, Equation (349) is hyperbolic and upon introducing the transformation

$$\xi = \lambda_1 x + y, \quad \eta = \lambda_2 x + y,$$

where λ_1 and λ_2 are roots of $A\lambda^2 + B\lambda + C = 0$, we find

$$\begin{aligned} A' &= 0, & D' &= 0, \\ B' &= -A(\lambda_1 - \lambda_2)^2, & E' &= 0, \\ C' &= 0, & F' &= 0. \end{aligned}$$

Hence Equation (349) becomes

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0,$$

whose general solution is

$$u = \phi(\xi) + \psi(\eta).$$

Returning to the original variables x and y , we see that the general solution of

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = 0,$$

where

$$B^2 - 4AC > 0,$$

* By *homogeneous* we now mean that all terms appearing in the equation have derivatives of the *same* order. In particular, Equation (349) is *homogeneous of the second order*. (It also happens to be homogeneous in our previous definition.)

is

$$u(x, y) = \phi(\lambda_1 x + y) + \psi(\lambda_2 x + y),$$

where ϕ and ψ are arbitrary functions, and λ_1 and λ_2 are the two distinct roots of the algebraic equation

$$A\lambda^2 + B\lambda + C = 0.$$

95. The parabolic equation

If the original Equation (331) is homogeneous in the second derivatives and parabolic,

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = 0, \quad (350)$$

it reduces to

$$\frac{\partial^2 u}{\partial \xi^2} = 0, \quad (351)$$

under the change of variables

$$\xi = x, \quad \eta = \lambda x + y,$$

where λ is the only real root of the algebraic equation $A\lambda^2 + B\lambda + C = 0$. The general solution of Equation (351) is therefore

$$u = \xi\phi(\eta) + \psi(\eta),$$

or, reverting to x and y , the most general solution of Equation (350) is

$$u(x, y) = x\phi(\lambda x + y) + \psi(\lambda x + y),$$

where ϕ and ψ are arbitrary functions, and λ is the only (real) root of the algebraic equation $A\lambda^2 + B\lambda + C = 0$.

96. The elliptic equation

We have seen (Section 91) that the general elliptic equation can be written

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a \frac{\partial u}{\partial x} + b \frac{\partial u}{\partial y} + cu = 0. \quad (352)$$

If we let

$$u(x, y) = w(x, y)e^{-\frac{1}{2}(ax+by)},$$

the derivatives of u are

$$\frac{\partial u}{\partial x} = \frac{\partial w}{\partial x} e^{-\frac{1}{2}(ax+by)} - \frac{a}{2} w e^{-\frac{1}{2}(ax+by)},$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 w}{\partial x^2} e^{-\frac{1}{2}(ax+by)} - a \frac{\partial w}{\partial x} e^{-\frac{1}{2}(ax+by)} + \frac{a^2}{4} w e^{-\frac{1}{2}(ax+by)},$$

$$\frac{\partial u}{\partial y} = \frac{\partial w}{\partial y} e^{-\frac{1}{2}(ax+by)} - \frac{b}{2} w e^{-\frac{1}{2}(ax+by)},$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 w}{\partial y^2} e^{-\frac{1}{2}(ax+by)} - b \frac{\partial w}{\partial y} e^{-\frac{1}{2}(ax+by)} + \frac{b^2}{4} w e^{-\frac{1}{2}(ax+by)}.$$

Substituting these expressions in Equation (352) yields

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \left(c - \frac{a^2}{4} - \frac{b^2}{4}\right)w = 0.$$

This equation has been met before in connection with the study of Mathieu functions.

If Equation (331) is homogeneous in the second derivatives and elliptic, that is, if

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = 0$$

and

$$B^2 - 4AC < 0,$$

the equation reduces to

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \tag{353}$$

under the change of variables

$$\xi = \lambda x + y, \quad \eta = \bar{\lambda} x + y,$$

where λ and $\bar{\lambda}$ are the complex conjugate roots of the algebraic equation $A\lambda^2 + B\lambda + C = 0$. Here ξ and η are complex variables. The general solution of Equation (353) is therefore

$$u(x, y) = \phi(\lambda x + y) + \psi(\bar{\lambda} x + y),$$

where ϕ and ψ are arbitrary functions.

PROBLEMS

Determine whether the following equations are hyperbolic, parabolic, or elliptic, and determine their general solution.

$$1. \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$$2. \quad 3 \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial x \partial y} + 7 \frac{\partial^2 u}{\partial y^2} = 0.$$

$$3. \quad 4 \frac{\partial^2 u}{\partial x^2} - 7 \frac{\partial^2 u}{\partial x \partial y} + 3 \frac{\partial^2 u}{\partial y^2} = 0.$$

$$4. \quad 4 \frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$$5. \quad a^2 \frac{\partial^2 u}{\partial x^2} + 2a \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$$

6. For what values of x and y is the following equation elliptic, parabolic, or hyperbolic?

$$x \frac{\partial^2 u}{\partial x^2} + x \frac{\partial^2 u}{\partial x \partial y} + y \frac{\partial^2 u}{\partial y^2} = 0.$$

7. Is the following equation hyperbolic, elliptic, or parabolic?

$$\frac{\partial^2 u}{\partial x^2} - 4 \frac{\partial^2 u}{\partial y^2} + x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} - 3xyu = 0.$$

8. Solve the equation of the vibrating string for the initial conditions

$$u(x, 0) = 0, \\ \frac{\partial}{\partial t} u(x, 0) = x(L - x).$$

9. In the derivation of the canonical form of the equations of elliptic type it was stated that $B' \neq 0$, and hence that one could divide the equation through by this quantity. Establish, in detail, that B' is different from zero.

10. Show in detail the statement made in Section 95 that if

$$A \frac{\partial^2 u}{\partial x^2} + B \frac{\partial^2 u}{\partial x \partial y} + C \frac{\partial^2 u}{\partial y^2} = 0$$

is parabolic, it reduces to $\frac{\partial^2 u}{\partial \xi^2} = 0$ under the change of variables

$$\xi = x, \quad \eta = \lambda x + y,$$

where $\lambda = -B/2A$.

11. From the general solution of the hyperbolic equation deduce the solution for the parabolic case as $\lambda_1 \rightarrow \lambda_2$.

[Hint : Consider the limit of

$$\frac{f(\lambda_1 x + y) - f(\lambda_2 x + y)}{\lambda_1 - \lambda_2}$$

as $\lambda_1 \rightarrow \lambda_2$.]

12. Find the solution of the parabolic equation

$$\frac{\partial^2 u}{\partial x^2} + 4 \frac{\partial^2 u}{\partial x \partial y} + 4 \frac{\partial^2 u}{\partial y^2} = 0$$

satisfying the boundary conditions

$$u(0, y) = \phi(y), \quad \frac{\partial}{\partial y} u(x, 0) = \psi(x).$$

13. Show that (Section 92)

$$\frac{\partial^2 w}{\partial x \partial y} + \alpha w = 0$$

reduces to a Bessel equation if we assume a solution of the form $w(x, y) = f(xy)$.

14. Write explicitly the most general linear partial differential equation of the second order in three independent variables.

15. Write explicitly the most general linear partial differential equation of the third order in two independent variables.

16. Find the general solution of

$$\sum_{r=0}^n a_r \frac{\partial^r u}{\partial x^r \partial y^{n-r}} = 0,$$

where the a_r are constants. Investigate the case of multiple roots.

REFERENCES

1. BATEMAN, H., *Partial Differential Equations of Mathematical Physics*, Dover Publications, 1944.
2. BOCHNER, S. and CHANDRASEKHARAN, K., *Fourier Transforms*, Princeton University Press, 1949.
3. BYERLY, W. E., *An Elementary Treatise on Fourier's Series and Spherical, Cylindrical, and Ellipsoidal Harmonics*, Ginn & Company, 1893.
4. CAMPBELL, G. A. and FOSTER, R. M., *Fourier Integrals for Practical Applications*, American Telephone & Telegraph Company, Printing of 1942.
5. CARSLAW, H. S., *Introduction to the Theory of Fourier's Series and Integrals*, Dover Publications, 3d. ed., 1930.
6. CHURCHILL, R. V., *Fourier Series and Boundary Value Problems*, McGraw-Hill Book Company, Inc., 1941.
7. FITE, W. B., *Advanced Calculus*, The Macmillan Company, 1938.
8. GUILLEMIN, E. A., *Communication Networks*, Vol. II, John Wiley & Sons, Inc., 1935.
9. HOBSON, E. W., *The Theory of Functions of a Real Variable and the Theory of Fourier's Series*, Vol. II, Cambridge University Press, 2d ed., 1926.
10. HODGMAN, C. D., Mathematical tables from *Handbook of Chemistry and Physics*, Chemical Rubber Publishing Company, 6th ed., 1938.
11. HOHEISEL, G., *Partielle Differentialgleichungen*, Walter de Gruyter & Company, Berlin and Leipzig, 1928.
12. INCE, E. L., *Ordinary Differential Equations*, Dover Publications, 1944.
13. INGERSOLL, L. R. and ZOBEL, O. J., *An Introduction to the Mathematical Theory of Heat Conduction*, Ginn & Company, 1913.
14. JAHNKE, E. and EMDE, F., *Tables of Functions with Formulae and Curves*, Dover Publications, 4th ed., 1945.
15. KELLOGG, O. D., *Foundations of Potential Theory*, Springer, Berlin, 1929.
16. LAMB, H., *Hydrodynamics*, Dover Publications, 1945.

17. LOVE, A. E. H., *A Treatise on the Mathematical Theory of Elasticity*, Cambridge University Press, 3d ed., 1920.
18. McLACHLAN, N. W., *Complex Variable and Operational Calculus with Technical Applications*, Cambridge University Press, 1944.
19. McLACHLAN, N. W., *Loud speakers: Theory, Performance, Testing and Design*, Oxford, Clarendon Press, 1934.
20. McLACHLAN, N. W., *Theory and Application of Mathieu Functions*, Oxford, Clarendon Press, 1947.
21. PAGE, L., *Introduction to Theoretical Physics*, D. Van Nostrand Company, Inc., 2d ed., 1935.
22. PEIRCE, B. O., *A Short Table of Integrals*, Ginn and Company, 3d ed., 1929.
23. SALVADORI, M. G., *Numerical Methods in Engineering*, Prentice-Hall, Inc., 1952.
24. SALVADORI, M. G. and MILLER, K. S., *The Mathematical Solution of Engineering Problems*, McGraw-Hill Book Company, Inc., 1948.
25. SALVADORI, M. G. and SCHWARZ, R. J., *Differential Equations in Engineering Problems*, Prentice-Hall, Inc., 1954.
26. STRATTON, J. A., *Electromagnetic Theory*, McGraw-Hill Book Company, Inc., 1st ed., 1941.
27. TIMOSHENKO, S., *Theory of Elastic Stability*, McGraw-Hill Book Company, Inc., 1st ed., 1936.
28. TIMOSHENKO, S., *Theory of Elasticity*, McGraw-Hill Book Company, Inc., 1934.
- ✓ 29. TIMOSHENKO, S., *Theory of Plates and Shells*, McGraw-Hill Book Company, Inc., 1st ed., 1940.
30. WATSON, G. N., *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, 2d ed., 1944.
31. WYLIE, C. R., *Advanced Engineering Mathematics*, McGraw-Hill Book Company, Inc., 1951.
32. ZYGMUND, A., *Trigonometrical Series*, Warszawa-Lwów, 1935.

INDEX

- Airy stress function, 36
- Beam theory, 15
- Bending of an elastic plate, 38
- Bernoulli-Euler theory, 15
- Bessel functions (*see* Bessel's equation)
- Bessel's equation, 179
 - application to heat flow, 209
 - functions, 202
 - graph of $J_0(x)$, 204
 - graph of $Y_0(x)$, 205
 - orthogonality of Bessel functions, 206
 - properties of Bessel functions, 202
 - solution of, 201
 - zeros of $J_0(x)$, 209
- Beta function, 226
- Biharmonic equation, 37
- Boundary conditions, 6
 - nonhomogeneous, 108
- Boundary value problems, 8
- Buckling of an elastic plate, 38
- Built-in beam, 19
- Canonical forms, 236
- Cauchy equation, 112, 115, 178
- Cauchy's problem, 241
- Characteristic functions, 223
- Characteristic numbers, 223
- Complex Fourier series, 76
- Cosine-elliptic functions, 213
- Cosine transform, 144
- Decomposition of material, 125
- Derivation of partial differential equations, 1
- Differential equations (*see* Bessel, Legendre, Mathieu, Ordinary, Partial)
- Differential system, 8
- Direct current transmission line, 27
- Dirichlet conditions, 83
- Distributed parameter system, 24
- Double Fourier series, 79
- Dynamical equations, 39
- Electrostatic potential, 29
- Elliptic coordinates, 180
- Elliptic partial differential equations, 240, 244
- Equations (*see also* Bessel, Indicial, Legendre, Mathieu, Nonhomogeneous, Ordinary differential, Partial differential);
 - of compatibility, 36
 - in three dimensions, 38
 - of continuity, 30
 - of elasticity, 36, 37
 - derivation of, 35
 - of equilibrium, 36
 - of motion of a fluid, 39
- Error function, 151
- Even functions, 56
- Exponential form of Fourier series (*see* Fourier series, complex form)
- Factorial, 186
- Flexural rigidity, 16, 38
- Flow of heat (*see* Heat flow)
- Force function, 28
- Forced vibrations, 125
- Fourier integral, 142, 153
 - cosine transform, 144, 147
 - examples, 155
 - inverse Fourier transform, 147
 - sine transform, 144, 147
 - spectrum, 147, 154
 - table of transforms, 160
 - transform of derivative, 161

- transform pairs, 155
- translation property, 159
- Fourier transform (*see* Fourier integral)
- Fourier series, 45
 - change of interval, 53
 - example, 54
 - coefficients, 47
 - complex form, 76
 - example, 78
 - Dirichlet conditions, 83
 - double, 79
 - example, 82
 - even and odd functions, 56
 - examples, 58
 - expansion in $(-L, L)$, 67
 - even and odd functions, 69
 - examples, 70
 - example, 68
 - prolongation of functions, 72
 - expansion in $(-\pi, \pi)$, 48
 - example, 50
 - formulas for expansion, 50
 - full Fourier interval, 50
 - half Fourier interval, 60
 - mathematical conditions on, 83
 - phase angles, 74
 - example, 75
 - points of discontinuity, 63
 - prolongation of functions, 60, 72
 - examples, 62
 - trial functions, 130
 - trigonometric approximation, 64
- Frobenius, method of, 190
- Functions,
 - Bessel, 202
 - beta, 226
 - cosine-elliptic, 213
 - even, 56
 - gamma, 187
 - Legendre, 194
 - Mathieu, 213
 - odd, 56
 - periodic, 45
 - sine-elliptic, 213
 - sine integral, 172
- Fundamental interval (*see* Fourier series, full Fourier interval)
- Gamma function, 187
 - graph of, 189
- Gravitational potential, 28
- Heat equation (*see* Heat flow)
- Heat flow;
 - Bessel's equation, 209
 - in a semi-circular plate, 112
 - in an infinite bar, 142
 - in one dimension, 20
 - assumptions, 20
 - derivation of equation of, 20
 - equation of, 22
 - in three dimensions, 22
 - derivation of equation of, 23
 - equation of, 23
 - in two dimensions, 24
 - solution by separation of variables, 101
 - solution by Fourier integral, 150
 - transform of heat equation, 161
- High-frequency line equations, 27
- Hooke's law, 12
- Hydrodynamics (*see* Equations of continuity)
- Hyperbolic partial differential equations, 238, 240, 243
- Impulse function, 157
- Indicial equation, 190
- Initial conditions, 6
- Inner product, 184
- Integrals, evaluation of, 149
- Inverse Fourier transform (*see* Fourier integral)
- Kirchhoff's laws, 25
- Kronecker delta, 46
- Laminar flow, 39
- Laplacian, 23
 - in polar coordinates, 113
 - in spherical coordinates, 176
- Legendre functions (*see* Legendre's equation)
- Legendre's equation, 178
 - application to potential, 199
 - generating function, 197

- orthogonality of Legendre polynomials, 195
- polynomials, 194
- Rodrigues' formula, 227
- solution of, 190
- Longitudinal vibrations of a bar;
 - assumptions, 12
 - derivation of equation of, 12
 - equation of, 14
- Lumped parameter system, 24
- Magnetic potential, 30
- Mathematical conditions;
 - on Fourier integral, 165
 - on Fourier series, 83
- Mathieu functions (*see* Mathieu's equation)
- Mathieu's equation, 183
 - application to vibrating membrane, 216
 - orthogonality of Mathieu functions, 215, 219
 - solution of, 212
- Maxwell's equations, 40
- Mean square error, 64
- Method of separation of variables, 93
- Nonhomogeneous boundary conditions (*see* Boundary conditions)
- Nonhomogeneous equations, 124
 - solution of, 126
- Odd functions, 56
- Ordinary differential equations, 1, 7, 27, 95, 178, 179, 183
- Orthogonal functions, 185
- Orthogonality conditions, 46
 - of Bessel functions, 206
 - of Legendre polynomials, 195
 - of Mathieu functions, 215, 219
- Orthonormal set, 185
 - complete, 186
- Oscillation in a loud speaker horn, 40
- Parabolic partial differential equations, 239, 244
- Partial differential equations;
 - derivations, 1
 - elliptic, 240, 244
 - hyperbolic, 238, 240, 243
 - parabolic, 239, 244
 - solution by Fourier integral, 150, 152
 - solution by separation of variables, 93
 - solution by transform, 161, 163
 - transformations of, 235
- Periodic functions, 45
- Phase angles, 74
- Poisson's ratio, 38
- Polar coordinates, 113
- Potential (*see also* Electrostatic, Gravitational, Magnetic);
 - Legendre's equation, 199
- Potential function, 28
- Power series (*see* Taylor series)
- Probability integral, 151
- Problems, 41, 84, 132, 166, 225, 246
- Radioactive decay, 124
 - solution of equation of, 126
- Separation of variables, 93
 - general case, 128
- Series (*see* Fourier, Taylor)
- Simply supported beam, 18
- Sine-elliptic functions, 213
- Sine integral function, 172
- Sine transform, 144
- Spectrum, 147
- Spherical coordinates, 176
- Square wave, 50
- Step function, 157
- Stress function, 36
- Sturm-Liouville theory, 222
- Submarine cable equations, 27
- Superposition of solutions, 97, 111
- Taylor series, 2, 45
- Telegraph equations, 24
- Temperature distribution in a slab (*see* Heat flow)
- Tidal waves, 32
- Torsion of a prismatical bar, 39
- Transform (*see* Fourier integral)

- Transformations of partial differential equations, 235
- Transmission line equations, 27
 - derivation of equations, 25
 - solution of, 105
- Transverse vibrations of a beam;
 - assumptions, 15
 - boundary conditions, 19
 - derivation of equation, 15
 - equation of, 18
 - solution of equation of, 117
- Traveling wave phenomenon, 99
- Trial functions, 128
 - based on Fourier series, 130
- Trigonometric approximations, 64
- Trigonometric identities, 66
- Velocity potential, 40
- Vibrating membrane;
 - assumptions, 8
 - boundary conditions, 12
 - derivation of equation of, 8
 - equation of, 11
 - Mathieu's equation, 216
 - solution of equation of, 121
- Vibrating string :
 - assumptions, 3
 - boundary conditions, 6
 - derivation of equation of, 3
 - equation of, 6
 - forced vibrations, 125
 - infinite in length, 152
 - solution by Fourier integral, 152
 - solution by separation of variables, 94
 - solution by transform, 163
 - solution in closed form, 98
- Vibration of a microphone diaphragm, 40
- Vibrations of a bar (*see* Longitudinal vibrations, Transverse vibrations of a beam)
- Voltage in a transmission line (*see* Transmission line equations)
- Wave equation, 6, 12, 99
 - Cauchy problem, 241
 - numerical example, 99
 - transform of, 163
- Zeros of Bessel function, 209
- Zonal harmonics, 194



✓

4. 11. 4. d. Dec. 10, 1964.



INST. SCI. TECH. LIB.
3115 I.S.T. BLDG., N.C.



